STAT-F-407
Continuous-time processes

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Outline of the course

Markov processes

A Markov process is a continuous-time Markov chain:

Let S be a finite or countable set (indexed by $i = 1, 2, ...$)
Let $(X_t)_{t \geq 0}$ be a SP with $X_t : (\Omega, \mathcal{A}, P) \to S$ for all t.

Definition: (X_t) is a homogeneous Markov process (HMP) on S
$\Leftrightarrow$

- (i) $\mathbb{P}[X_{t+s} = j | X_u, 0 \leq u \leq t] = \mathbb{P}[X_{t+s} = j | X_t] \quad \forall t, s \, \forall j.$
- (ii) $\mathbb{P}[X_{t+s} = j | X_t = i] = \mathbb{P}[X_s = j | X_0 = i] \quad \forall t, s \, \forall i, j.$

Remarks:

- (i) is the Markov property, whereas (ii) is related to time-homogeneity.
- (ii) allows for defining the transition functions $p_{ij}(s) = \mathbb{P}[X_{t+s} = j | X_t = i]$

Markov processes

Further remarks:

- As for Markov chains, we will collect the transition functions $p_{ij}(s)$ in the transition matrices $P(s) = (p_{ij}(s))$.
- Those transition matrices $P(s)$ are stochastic for all s, i.e., $p_{ij}(s) \in [0, 1]$ for all i, j and $\sum_j p_{ij}(s) = 1$ for all i.
- The Chapman-Kolmogorov equations now state that

  $$P(t + s) = P(t)P(s)$$

  that is,

  $$\mathbb{P}[X_{t+s} = j | X_0 = i] = \sum_k \mathbb{P}[X_t = k | X_0 = i]\mathbb{P}[X_s = j | X_0 = k],$$

  for all s, t and all i, j (exercise).

Markov processes

Let $W_t = \inf\{s > 0 \mid X_{t+s} \neq X_t\}$ be the survival time of state X_t from t.

$\rightsquigarrow$ **Theorem**: *let $i \in S$. Then either*

- *(i) $W_t | [X_t = i] = 0$ a.s., or*
- *(ii) $W_t | [X_t = i] = \infty$ a.s., or*
- *(iii) $W_t | [X_t = i] \sim \mathrm{Exp}(\lambda_i)$ for some $\lambda_i > 0$.*

Markov processes

Proof:

Let $f_i(s) := \mathbb{P}[W_t > s | X_t = i] = \mathbb{P}[W_0 > s | X_0 = i]$ (by homogeneity). Then, for all $s_1, s_2 > 0$,

$$f_i(s_1 + s_2) = \mathbb{P}[W_0 > s_1 + s_2 | X_0 = i] = \mathbb{P}[W_0 > s_1, W_{s_1} > s_2 | X_0 = i]$$
$$= \mathbb{P}[W_{s_1} > s_2 | W_0 > s_1, X_0 = i]\mathbb{P}[W_0 > s_1 | X_0 = i]$$
$$= \mathbb{P}[W_{s_1} > s_2 | X_{s_1} = i] f_i(s_1) = f_i(s_1) f_i(s_2).$$

Assume that $\exists s_0 > 0$ such that $f_i(s_0) > 0$ (if this is not the case, (i) holds). Then

- $0 < f_i(s_0) = f_i(s_0 + 0) = f_i(s_0) f_i(0)$, so that $f_i(0) = 1$.
- Now,

$$f_i'(s) = \lim_{h \to 0} \frac{f_i(s+h) - f_i(s)}{h} = f_i(s) \lim_{h \to 0} \frac{f_i(h) - f_i(0)}{h} = f_i(s) f_i'(0).$$

Therefore, letting $\lambda_i := -f_i'(0)$,

$$\frac{f_i'(s)}{f_i(s)} = -\lambda_i,$$

so that

$$\ln f_i(s) = \ln f_i(s) - \ln f_i(0) = \int_0^s \frac{f_i'(u)}{f_i(u)}\, du = \int_0^s (-\lambda_i)\, du = -\lambda_i s,$$

for all $s > 0$. Hence,

$$f_i(s) = \mathbb{P}[W_t > s | X_t = i] = \exp(-\lambda_i s),$$

which establishes the result (note that (ii) corresponds to the case $\lambda_i = 0$). $\square$

Markov processes

Theorem: *Let $i \in S$. Then either*

- *(i)* $W_t | [X_t = i] = 0$ *a.s., or*
- *(ii)* $W_t | [X_t = i] = \infty$ *a.s., or*
- *(iii)* $W_t | [X_t = i] \sim \mathrm{Exp}(\lambda_i)$ *for some* $\lambda_i > 0$.

This result leads to the following classification of states:

- In case (i), *i* is said to be instantaneous (as soon as the process goes to *i*, it goes away from it).
- In case (ii), *i* is said to be absorbant (if the process goes to *i*, it remains there forever).
- In case (iii), *i* is said to be stable (if the process goes to *i*, it remains there for some exponentially distributed time).

Assume that (X_t) is conservative (i.e. there is no instantaneous state). Then a typical sample path is

Associate with (X_t) both following SP:

- (a) the process of survival times $(T_{n+1} - T_n)_{n\in\mathbb{N}}$, where $T_0 = 0$ and $T_{n+1} = T_n + W_{T_n}$, $n \in \mathbb{N}$;
- (b) the jump chain $(\tilde{X}_n)_{n\in\mathbb{N}}$, where $\tilde{X}_n = X_{T_n}$, $n \in \mathbb{N}$.

Theorem: *Assume that* (X_t) *is conservative. Then*

$$\mathbb{P}[\tilde{X}_{n+1} = j,\ T_{n+1} - T_n > s \,|\, \tilde{X}_0 = i_0, \ldots, \tilde{X}_n = i_n, T_1, \ldots, T_n]$$
$$= \mathbb{P}[\tilde{X}_{n+1} = j,\ T_{n+1} - T_n > s \,|\, \tilde{X}_n = i_n] = e^{-\lambda_{i_n} s}\, \tilde{P}_{i_n j},$$

where $\tilde{P} = (\tilde{P}_{ij})$ *is the transition matrix of a Markov chain such that*

$$\tilde{P}_{ii} = \begin{cases} 0 & \textit{if } i \textit{ is stable} \\ 1 & \textit{if } i \textit{ is absorbant.} \end{cases}$$

This shows that

- (a) the jump chain is a HMC and
- (b) conditionally on $\tilde{X}_0, \ldots, \tilde{X}_n$, the survival times $T_{n+1} - T_n$ are independent.

Markov processes

If (X_t) is a conservative HMP, we can determine

- the process of survival times $(T_{n+1} - T_n)_{n \in \mathbb{N}}$ and
- the jump chain $(\tilde{X}_n)_{n \in \mathbb{N}}$.

One might ask whether it is possible to go the other way around, that is, to determine (X_t) from

- the process of survival times $(T_{n+1} - T_n)_{n \in \mathbb{N}}$ and
- the jump chain $(\tilde{X}_n)_{n \in \mathbb{N}}$.

The answer:

Yes, provided that (X_t) is regular, that is, is such that

$$\lim_{n \to \infty} T_n = \infty.$$

Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
4. Markov chains.
5. Markov processes, Poisson processes.
   5.1. Markov processes.
   5.2. Poisson processes.
6. Brownian motions.

Poisson processes

Definition: $(N_t = X_t)$ is a Poisson process (with parameter $\lambda > 0$) $\Leftrightarrow$ (X_t) is a regular HMP, for which $S = \mathbb{N}$,

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & 0 & 1 & 0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

and

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \vdots \end{pmatrix}.$$

Poisson processes

A typical sample path:

Poisson processes

Remarks:

- (i) The survival times $W_n := W_{T_{n-1}}$ ($n \in \mathbb{N}_0$) are i.i.d. $\mathrm{Exp}(\lambda)$.
- (ii) $T_n = \sum_{i=1}^n W_i$ has an Erlang distribution with parameters n and λ, that is,

$$F^{T_n}(t) = \mathbb{P}[T_n \leq t] = \begin{cases} 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

- (iii) For all $t > 0$, $N_t \sim \mathcal{P}(\lambda t)$. Indeed,

$$\mathbb{P}[N_t \leq k] = \mathbb{P}[T_{k+1} > t] = \sum_{i=0}^{k} \frac{(\lambda t)^i}{i!} e^{-\lambda t},$$

  so that $\mathbb{P}[N_t = k] = \mathbb{P}[N_t \leq k] - \mathbb{P}[N_t \leq k-1] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$.
- (iv) Hence, $\mathbb{E}[W_n] = 1/\lambda$ and $\mathbb{E}[N_t] = \lambda t$ ($\rightsquigarrow \lambda$ is a rate).

Poisson processes

How to check (ii)?

- $T_1 = W_1 \sim \mathrm{Exp}(\lambda)$, so that (ii) holds true for $n = 1$.
- It remains to show that if (ii) holds for n, it also holds for $n+1$, which can be achieved in the following way:

$$
\begin{aligned}
F^{T_{n+1}}(t) &= 1 - \mathbb{P}[T_{n+1} > t] = 1 - \int_0^\infty \mathbb{P}[T_{n+1} > t | T_n = u] f^{T_n}(u)\, du \\
&= 1 - \int_0^t \mathbb{P}[T_{n+1} > t | T_n = u] f^{T_n}(u)\, du \\
&- \int_t^\infty \mathbb{P}[T_{n+1} > t | T_n = u] f^{T_n}(u)\, du \\
&= 1 - \int_0^t \mathbb{P}[W_{n+1} > t - u] f^{T_n}(u)\, du - \int_t^\infty f^{T_n}(u)\, du \\
&= 1 - \int_0^t e^{-\lambda(t-u)} f^{T_n}(u)\, du - \Big(F^{T_n}(\infty) - F^{T_n}(t)\Big) = \ldots
\end{aligned}
$$

Poisson processes

An important feature of Poisson processes:

Theorem: *for all* t, h *and* k*,*

$$\mathbb{P}[N_{t+h} - N_t = k \mid N_u,\, 0 \le u \le t] = e^{-\lambda h} \frac{(\lambda h)^k}{k!}.$$

Proof: From the Markov property,

$$\mathbb{P}[N_{t+h} - N_t = k \mid N_u,\, 0 \le u \le t] = \mathbb{P}[N_{t+h} - N_t = k \mid N_t].$$

Now, $\mathbb{P}[N_{t+h} - N_t = k \mid N_t = n] = ?$

Consider the SP $(\tilde{N}_h := N_{t+h} - N_t = N_{t+h} - n \,|\, h \geq 0)$, with survival times $\tilde{W}_1, \tilde{W}_2, \ldots$, say.

Poisson processes

Clearly,

- the jump chain of $(\tilde{N}_h)$ is that of a Poisson process, and
- $\tilde{W}_2, \tilde{W}_3, \ldots$ are i.i.d. $\text{Exp}(\lambda)$.

As for $\tilde{W}_1$ (that is clearly independent of the other $\tilde{W}_i$'s),

$$\mathbb{P}[\tilde{W}_1 > w] = \mathbb{P}[W_{n+1} > \Delta + w \,|\, W_{n+1} > \Delta]$$
$$= \mathbb{P}[W_{n+1} > \Delta + w]/\mathbb{P}[W_{n+1} > \Delta] = e^{-\lambda(\Delta+w)}/e^{-\lambda\Delta} = e^{-\lambda w},$$

for all $w > 0$, so that $\tilde{W}_1 \sim \text{Exp}(\lambda)$.

Hence, $(\tilde{N}_h)$ is a Poisson process, and we have

$$\mathbb{P}[N_{t+h} - N_t = k \,|\, N_t = n] = \mathbb{P}[\tilde{N}_h = k] = e^{-\lambda h} \frac{(\lambda h)^k}{k!}, \quad \forall k.$$

$\square$

Poisson processes

Theorem: *for all t, h and k,*

$$\mathbb{P}[N_{t+h} - N_t = k \mid N_u,\, 0 \leq u \leq t] = e^{-\lambda h}\frac{(\lambda h)^k}{k!}.$$

This result implies that

- (i) if $0 = t_0 < t_1 < t_2 < \ldots$, the $N_{t_{i+1}} - N_{t_i}$'s are independent.
- (ii) $N_{t_{i+1}} - N_{t_i} \sim \mathcal{P}(\lambda(t_{i+1} - t_i))$ (stationarity of the increments).

Part (ii) shows that

$$\mathbb{P}[k \text{ events in } [t, t+h)] = \begin{cases} 1 - \lambda h + o(h) & \text{if } k = 0 \\ \lambda h + o(h) & \text{if } k = 1 \\ o(h) & \text{if } k \geq 2. \end{cases}$$

Compound Poisson processes

Let $(N_t)_{t\geq 0}$ be a Poisson process.
Let Y_k, $k \in \mathbb{N}_0$ be positive i.i.d. r.v.'s (independent of (N_t)).

Definition: $(S_t)_{t\geq 0}$ is a compound Poisson process

$$\Updownarrow$$

$$S_t = \begin{cases} 0 & \text{if } N_t = 0 \\ \sum_{k=1}^{N_t} Y_k & \text{if } N_t \geq 1. \end{cases}$$

This SP plays a crucial role in the most classical model in actuarial sciences...

Compound Poisson processes

Denoting by Z_t the wealth of an insurance company at time t, this model is

$$Z_t = u + c\,t - S_t,$$

where

- u is the initial wealth,
- c is the "income rate" (determining the premium), and
- $S_t = \left(\sum_{k=1}^{N_t} Y_k\right)\mathbb{I}_{[N_t \geq 1]}$ is a compound Poisson process that models the costs of all sinisters up to time t (there are N_t sinisters, with random costs $Y_1, Y_2, \ldots, Y_{N_t}$ for the company up to time t).

Compound Poisson processes

A typical sample path:

Compound Poisson processes

Let $T = \inf\{t > 0 \,|\, Z_t < 0\}$ be the time at which the company goes bankrupt.

Let $\psi(u) = \mathbb{P}[T < \infty \,|\, Z_0 = u]$ be the ruin probability (when starting from $Z_0 = u$).

Then one can show the following:

Theorem: *assume $\mu = \mathbb{E}[Y_k] < \infty$. Denote by λ the parameter of the underlying Poisson process. Then,*

- *(i) if $c \leq \lambda\mu$, $\psi(u) = 1$ for all $u > 0$;*
- *(ii) if $c > \lambda\mu$, $\psi(u) < 1$ for all $u > 0$.*

This shows that if it does not charge enough, the company will go bankrupt a.s.

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    6.1. Definition.
    6.2. Markov property, martingales.
    6.3. Gaussian processes and Brownian bridges.

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A heuristic introduction

Consider a symmetric RW (starting from 0) $X_n = \sum_{i=1}^{n} Y_i$, where the Y_i's are i.i.d. with $\mathbb{P}[Y_i = 1] = \mathbb{P}[Y_i = -1] = \frac{1}{2}$. Now, assume that, at each Δt units of time, we make a step with length Δx. Then, writing $n_t = \lfloor t/(\Delta t) \rfloor$,

$$X_t = (\Delta x) \sum_{i=1}^{n_t} Y_i,$$

where we consider $(X_t)_{t \geq 0}$ as a continuous-time SP.

Our goal is to let $\Delta x, \Delta t \to 0$ in such a way we obtain a non-trivial limiting process. This requires a non-zero bounded limiting value of

$$\mathrm{Var}[X_t] = (\Delta x)^2 \mathrm{Var}\Big[\sum_{i=1}^{n_t} Y_i\Big] = (\Delta x)^2 \sum_{i=1}^{n_t} \mathrm{Var}[Y_i] = (\Delta x)^2 n_t,$$

which leads to the choice $\Delta x = \sigma\sqrt{\Delta t}$; the resulting variance is then $\sigma^2 t$ (note that we always have $\mathbb{E}[X_t] = 0$).

A heuristic introduction

What are the properties of the limiting process $(X_t)_{t\geq 0}$?

$$X_t = \lim_{\Delta t \to 0} \sigma \sqrt{\Delta t} \sum_{i=1}^{n_t} Y_i$$

- $X_0 = 0$.
- X_t is the limit of a sum of i.i.d. r.v.'s properly normalized so that $\mathbb{E}[X_t] = 0$ and $\mathrm{Var}[X_t] = \sigma^2 t$. Hence, $X_t \sim \mathcal{N}(0, \sigma^2 t)$;
- for each RW, the "increments" in disjoint time intervals are $\perp\!\!\!\perp$. $\rightsquigarrow$ This should also hold in the limit, i.e.,
  $\forall 0 \leq t_1 < t_2 < \ldots < t_k, \quad X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_k} - X_{t_{k-1}} \text{ are } \perp\!\!\!\perp;$
- for each RW, the increments are stationary (that is, their distribution in $[k, k+n]$ does not depend on k). $\rightsquigarrow$ This should also hold in the limit, i.e.
  $\forall s, t > 0, \quad X_{t+s} - X_t \overset{\mathcal{D}}{=} X_s - X_0.$

Definition

This leads to the following definition:

Definition: the SP $(X_t)_{t \geq 0}$ is a Brownian motion $\Leftrightarrow$

- $X_0 = 0$.
- for all $t > 0$, $X_t \sim \mathcal{N}(0, \sigma^2 t)$;
- the increments in disjoint time-intervals are $\perp\!\!\!\perp$, i.e.

  $$\forall 0 \leq t_1 < t_2 < \ldots < t_k, \quad X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_k} - X_{t_{k-1}} \text{ are } \perp\!\!\!\perp;$$

- the increments in equal-length time-intervals are stationary, i.e.

  $$\forall s, t > 0, \quad X_{t+s} - X_t \stackrel{\mathcal{D}}{=} X_s - X_0;$$

- the sample paths of $(X_t)_{t \geq 0}$ are a.s. continuous.

Definition

A typical sample path:

It can be shown that the sample paths (a.s.) are nowhere differentiable...

Definition

Remarks:

- Also called a Wiener Process (this type of SP was first studied rigourously by Wiener in 1923. It was used earlier by Brown and Einstein as a model for the motion of a small particle immersed in a liquid or a gas, and hence subject to mollecular collisions).
- If $\sigma = 1$, (X_t) is said to be standard. Clearly, if σ is known, one can always assume the underlying process is standard.
- Sometimes, one also includes a drift in the model $\rightsquigarrow (X_t := \mu t + \sigma B_t)$, where B_t a standard BM.
- In finance, μ is the trend and σ is the volatility.

Definition

A typical sample path:

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BM and the Markov property

Using the independence between disjoint increments, we straightforwardly obtain

$$\mathbb{P}[X_{t+s} \in B \,|\, X_u, 0 \leq u \leq t] = \mathbb{P}[X_{t+s} \in B \,|\, X_t].$$

This is nothing but the Markov property.

Also, note that

$$\mathbb{P}[X_{t+s} \in B \,|\, X_t = x] = \mathbb{P}[X_{t+s} - X_t \in B - x \,|\, X_t - X_0 = x]$$
$$= \mathbb{P}[X_{t+s} - X_t \in B - x] = \mathbb{P}[X_{t+s} - X_t + x \in B] = \mathbb{P}[Y \in B],$$

where $Y \sim \mathcal{N}(x, s)$. Hence,

$$\mathbb{P}[X_{t+s} \in B \,|\, X_t = x] = \int_B \frac{1}{\sqrt{2\pi s}} e^{-(y-x)^2/(2s)}\, dy.$$

BM and Martingales

Continuous-time martingales are defined in a similar way as for discrete-time ones. More precisely:

The SP $(M_t)_{t\geq 0}$ is a martingale w.r.t. the filtration $(\mathcal{A}_t)_{t\geq 0}$ $\Leftrightarrow$

- (i) $(M_t)_{t\geq 0}$ is adapted to $(\mathcal{A}_t)_{t\geq 0}$.
- (ii) $\mathbb{E}[|M_t|] < \infty$ for all t.
- (iii) $\mathbb{E}[M_t|\mathcal{A}_s] = M_s$ a.s. for all $s < t$.

Proposition: *let (X_t) be a standard BM. Then*

- *(a)* $(X_t)_{t\geq 0}$,
- *(b)* $(X_t^2 - t)_{t\geq 0}$, *and*
- *(c)* $\{e^{\theta X_t - \frac{\theta^2 t}{2}})_{t\geq 0}$

are martingales w.r.t. $\mathcal{A}_t = \sigma(X_u, 0 \leq u \leq t)$

Proof: in each case, (i) is trivial and (ii) is left as an exercise.
As for (iii):

(a) $\mathbb{E}[X_t|\mathcal{A}_s] = \mathbb{E}[X_s|\mathcal{A}_s] + \mathbb{E}[X_t - X_s|\mathcal{A}_s] = X_s + \mathbb{E}[X_t - X_s] = X_s$.

(b)

$$\begin{aligned}
\mathbb{E}[X_t^2 - t|\mathcal{A}_s] &= \mathbb{E}[(X_s + (X_t - X_s))^2|\mathcal{A}_s] - t \\
&= X_s^2 + 2X_s\mathbb{E}[X_t - X_s|\mathcal{A}_s] + \mathbb{E}[(X_t - X_s)^2|\mathcal{A}_s] - t \\
&= X_s^2 + 2X_s\mathbb{E}[X_t - X_s] + \mathbb{E}[(X_t - X_s)^2] - t \\
&= X_s^2 + \mathrm{Var}[X_t - X_s] - t \\
&= X_s^2 + (t - s) - t \\
&= X_s^2 - s.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{E}\left[e^{\theta X_t - \frac{\theta^2 t}{2}} | \mathcal{A}_s\right] &= e^{\theta X_s - \frac{\theta^2 t}{2}} \mathbb{E}\left[e^{\theta(X_t - X_s)} | \mathcal{A}_s\right] \\
&= e^{\theta X_s - \frac{\theta^2 t}{2}} \mathbb{E}\left[e^{\theta(X_t - X_s)}\right] \\
&= e^{\theta X_s - \frac{\theta^2 t}{2}} \mathbb{E}\left[e^{\theta\sqrt{t-s}Z}\right],
\end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$. But

$$\begin{aligned}
\mathbb{E}\left[e^{\theta\sqrt{t-s}Z}\right] &= \int_{\mathbb{R}} e^{\theta\sqrt{t-s}z} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \, dz \\
&= e^{\frac{\theta^2(t-s)}{2}} \int_{\mathbb{R}} \frac{e^{-\frac{(z-\theta\sqrt{t-s})^2}{2}}}{\sqrt{2\pi}} \, dz = e^{\frac{\theta^2(t-s)}{2}},
\end{aligned}$$

which yields the result. □

The optional stopping theorem (OST) still holds in this continuous-time setup, yielding results such as the following:

Proposition: *let (X_t) be a standard BM. Fix $a, b > 0$. Define $T_{ab} := \inf\{t > 0 \ : X_t \notin (-a, b)\}$. Then*

- *(i)* $\mathbb{E}[X_{T_{ab}}] = 0$,
- *(ii)* $\mathbb{P}[X_{T_{ab}} = -a] = \frac{b}{a+b}$, $\mathbb{P}[X_{T_{ab}} = b] = \frac{a}{a+b}$, *and*
- *(iii)* $\mathbb{E}[T_{ab}] = ab$.

Proof: (i) this follows from the OST and the fact (X_t) is a martingale.
(ii) $0 = \mathbb{E}[X_{T_{ab}}] = (-a) \times \mathbb{P}[X_{T_{ab}} = -a] + b \times (1 - \mathbb{P}[X_{T_{ab}} = -a])$.
Solving for $\mathbb{P}[X_{T_{ab}} = -a]$ yields the result.
(iii) The OST and the fact $(X_t^2 - t)$ is a martingale imply that $\mathbb{E}[X_{T_{ab}}^2 - T_{ab}] = \mathbb{E}[X_0^2 - 0] = 0$, which yields
$(-a)^2 \times \frac{b}{a+b} + b^2 \times \frac{a}{a+b} - \mathbb{E}[T_{ab}] = 0$. □

As for the martingale $\left(e^{\theta X_t - \frac{\theta^2 t}{2}}\right)_{t \geq 0}$, it allows for establishing results such as the following:

Proposition: *let (X_t) be a standard BM. Fix $c, d > 0$. Then* $\mathbb{P}[X_t \geq ct + d \textit{ for some } t \geq 0] = e^{-2cd}$.

Outline of the course

BM and Gaussian processes

Let (X_t) be a SP.

Definition: (X_t) is a Gaussian process $\Leftrightarrow$ for all k, for all $t_1 < t_2 < \ldots < t_k$, $(X_{t_1}, \ldots, X_{t_k})'$ is a Gaussian r.v.

Remark: the distribution of a Gaussian process is completely determined by

- its mean function $t \mapsto \mathbb{E}[X_t]$ and
- its autocovariance function $(s, t) \mapsto \mathrm{Cov}[X_s, X_t]$.

Proposition: A standard BM (X_t) is a Gaussian process with mean function $t \mapsto \mathbb{E}[X_t] = 0$ and autocovariance function $(s, t) \mapsto \mathrm{Cov}[X_s, X_t] = \min(s, t)$.

This might also be used as an alternative definition for BMs...

Proof: let (X_t) be a standard BM.

(i) For $s < t$, $X_t - X_s \stackrel{\mathcal{D}}{=} X_{t-s} - X_{s-s} = X_{t-s} \sim \mathcal{N}(0, t-s)$.

By using the independence between disjoint increments, we obtain, for $0 =: t_0 < t_1 < t_2 < \ldots < t_k$,

$$\begin{pmatrix} X_{t_1} - X_0 \\ X_{t_2} - X_{t_1} \\ \vdots \\ X_{t_k} - X_{t_{k-1}} \end{pmatrix} \sim \mathcal{N}(0, \Lambda),$$

where $\Lambda = (\lambda_{ij})$ is diagonal with $\lambda_{ii} = t_i - t_{i-1}$.

BM and Gaussian processes

Hence,

$$\sum_{i=1}^{k} v_i X_{t_i} = v' \begin{pmatrix} X_{t_1} \\ X_{t_2} \\ \vdots \\ X_{t_k} \end{pmatrix} = v' \begin{pmatrix} 1 & 0 & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} X_{t_1} - X_0 \\ X_{t_2} - X_{t_1} \\ \vdots \\ X_{t_k} - X_{t_{k-1}} \end{pmatrix}$$

is normally distributed, so that (X_t) is a Gaussian process.

(ii) Clearly, $t \mapsto \mathbb{E}[X_t] = 0$ for all t.

(iii) Eventually, assuming that $s < t$, we have

$$\mathrm{Cov}[X_s, X_t] = \mathrm{Cov}[X_s, X_s + (X_t - X_s)] = \mathrm{Var}[X_s] + \mathrm{Cov}[X_s, X_t - X_s] =$$

$$= s + \mathrm{Cov}[X_s - X_0, X_t - X_s] = s + 0 = \min(s, t).$$

$\square$

Brownian bridges

Let $(X_t)_{t\geq 0}$ be a BM.

Definition: if (X_t) is a BM, $(X_t - tX_1)_{0\leq t\leq 1}$ is a Brownian bridge.

Alternatively, it can be defined as a Gaussian process (over $(0,1)$) with mean function $t \mapsto \mathbb{E}[X_t] = 0$ and autocovariance function $(s,t) \mapsto \mathrm{Cov}[X_s, X_t] = \min(s,t)(1-\max(s,t))$ (exercise).

Application:

Let $X_1, \ldots, X_n$ be i.i.d. with cdf F.
Let $F_n(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{[X_i \leq x]}$ be the empirical cdf.

The LLN implies that $F_n(x) \stackrel{a.s.}{\to} \mathbb{E}[\mathbb{I}_{[X_1\leq x]}] = F(x)$ as $n \to \infty$.
Actually, it can be shown that $\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \stackrel{a.s.}{\to} 0$ as $n \to \infty$ (Glivenko-Cantelli theorem).

Brownian bridges

Assume that $X_1, \ldots, X_n$ are i.i.d. $\mathrm{Unif}(0,1)$ ($F(x) = x\mathbb{I}_{[x\in[0,1]]} + \mathbb{I}_{[x>1]}$).
Let $U_n(x) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\mathbb{I}_{[X_i \leq x]} - x\right)$, $x \in [0,1]$.

Then it can be shown that, as $n \to \infty$,

$$\sup_{x\in[0,1]} |U_n(x)| \xrightarrow{\mathcal{D}} \sup_{x\in[0,1]} |U(x)|,$$

where $(U(x))_{0\leq x\leq 1}$ is a Brownian bridge (Donsker's theorem).
Coming back to the setup where $X_1, \ldots, X_n$ are i.i.d. with (unknown) cdf F, the result above allows for testing

$$\begin{cases} \mathcal{H}_0: & F = F_0 \\ \mathcal{H}_1: & F \neq F_0, \end{cases}$$

where F_0 is some fixed (continuous) cdf.

Brownian bridges

The so-called Kolmogorov-Smirnov test consists in rejecting $\mathcal{H}_0$ if the value of

$$\sup_{x\in[0,1]} |U_n(x)| := \sup_{x\in[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\mathbb{I}_{[F_0(X_i)\leq x]} - x \right) \right|$$

exceeds some critical value (that is computed from Donsker's theorem).
This is justified by the fact that, under $\mathcal{H}_0$, $F_0(X_1), \ldots, F_0(X_n)$ are i.i.d. $\mathrm{Unif}(0,1)$ (exercise).