

# STAT-F-407

## Martingales

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## Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
4. Markov chains.
5. Markov processes, Poisson processes.
6. Brownian motions.

## Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
  - 3.1. Conditional expectations.
  - 3.2. Definitions and examples.
  - 3.3. Stopping times and the optional stopping theorem.
  - 3.4. Sub- and super- martingales, gambling strategies.
  - 3.5. Limiting results.
4. Markov chains.
5. Markov processes, Poisson processes.
6. Brownian motions.

## Independence

Independence can be extended to  $\sigma$ -algebras:

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Let  $\mathcal{A}_1, \mathcal{A}_2, \dots \subset \mathcal{A}$  be  $\sigma$ -algebras.

- ▶  $\mathcal{A}_1 \perp\!\!\!\perp \mathcal{A}_2 \stackrel{\text{def}}{\Leftrightarrow} \forall A_1 \in \mathcal{A}_1, \forall A_2 \in \mathcal{A}_2, A_1 \perp\!\!\!\perp A_2$ .
- ▶  $\mathcal{A}_1, \dots, \mathcal{A}_n$  are  $\perp\!\!\!\perp \stackrel{\text{def}}{\Leftrightarrow} \forall A_1 \in \mathcal{A}_1, \dots, \forall A_n \in \mathcal{A}_n, A_1, \dots, A_n$  are  $\perp\!\!\!\perp$ .
- ▶  $\mathcal{A}_1, \mathcal{A}_2, \dots$  are  $\perp\!\!\!\perp \stackrel{\text{def}}{\Leftrightarrow}$  for every  $n \geq 2, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$  are  $\perp\!\!\!\perp$ .

## Independence

Denote by

$$\sigma(X) \stackrel{\text{def}}{=} X^{-1}(\mathcal{B})$$

the  $\sigma$ -algebra generated by the r.v.  $X$ .

- ▶ Then  $X_1 \perp\!\!\!\perp X_2 \Leftrightarrow \sigma(X_1) \perp\!\!\!\perp \sigma(X_2)$ .
- ▶ This also allows for defining independence between r.v. and  $\sigma$ -algebras:

$$X_1, \dots, X_n, \mathcal{A}_1, \dots, \mathcal{A}_n \text{ are } \perp\!\!\!\perp$$



$$\sigma(X_1), \dots, \sigma(X_n), \mathcal{A}_1, \dots, \mathcal{A}_n \text{ are } \perp\!\!\!\perp.$$

## Conditional expectation

Conditional expectation allows for exploiting some information (e.g., the information that some event occurred) in order to improve the (unconditional) best guess  $\mathbb{E}[X]$ .

This available information can take various forms:

- ▶ (the occurrence of) an event.
- ▶ (the value of ) a r.v.
- ▶ (the occurrence of some event in) a  $\sigma$ -algebra.

## Radon-Nikodym

Let  $\mu$  and  $\nu$  be two measures over  $(\Omega, \mathcal{A})$ . We say that  $\nu$  is absolutely continuous with respect to  $\mu$  if for any  $A \in \mathcal{A}$ ,  $\mu(A) = 0$  implies  $\nu(A) = 0$

The following Theorem plays a central role in the definition of conditional expectations.

**Theorem:** Let  $\mu$  and  $\nu$  be two measures over  $(\Omega, \mathcal{A})$  with  $\mu$  being  $\sigma$ -finite. Then  $\nu$  is absolutely continuous with respect to  $\mu$  iff there exists a nonnegative function  $f : \Omega \rightarrow \mathbb{R}^+$  such that for all  $A \in \mathcal{A}$ ,

$$\nu(A) = \int_A d\nu = \int_A f d\mu \quad (1)$$

## Radon-Nikodym

The function  $f$  in the RN Theorem is *essentially unique* in the sense that if  $f_1$  and  $f_2$  are two functions such that (1) hold, then  $f_1$  and  $f_2$  are equal up to  $\mu$ -measure zero sets.

The set of  $\mu$ -almost everywhere equal functions is often denoted as  $\frac{d\nu}{d\mu}$  and is called the *Radon-Nikodym derivative* of  $\nu$  with respect to  $\mu$ .



## Conditional expectation

Let  $X$  be an integrable r.v. on  $(\Omega, \mathcal{A}, \mathbb{P})$ . Consider another real random variable  $Y$  defined over  $(\Omega, \mathcal{A})$  and denote by

$$P^Y(B) := P(Y^{-1}(B)),$$

$B \in \mathcal{B}$  the probability measure induced by  $Y$  (the distribution of  $Y(X)$ )

A measurable function  $g : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}, \mathcal{B})$  is said to be  $Y$ -measurable if there exists a measurable mapping  $h : (\mathbb{R}, \mathcal{B}) \rightarrow (\mathbb{R}, \mathcal{B})$  such that  $g(\omega) = h(Y(\omega))$ . If such a function exists, we have that

$$\int_B h(y) P^Y(y) = \int_{Y^{-1}(B)} h(Y(x)) dP(x) = \int_{Y^{-1}(B)} g(x) dP(x),$$

## Conditional expectation

This allows to compute integrals of  $Y$ -measurable functions in  $\Omega$ . Can this be extended to functions  $g$  that are not  $Y$ -measurables. This is the purpose of conditional expectations.

Let  $g$  be a non-negative,  $P$ -integrable function. Can we define a  $Y$ -measurable function  $h$  such that

$$\int_B h(y) P^Y(y) = \int_{Y^{-1}(B)} g(x) dP(x)?$$

## Conditional expectation

Since  $g$  is non-negative and  $P$ -integrable,

$$\nu_g(B) := \int_{Y^{-1}(B)} g(x) dP(x)$$

is a finite measure over  $(\mathbb{R}, \mathcal{B})$ . This measure  $\nu_g$  is absolutely continuous with respect to  $P^Y$  (since  $P^Y(B) = 0$  implies  $P(Y^{-1}(B)) = 0$ ). The RN Theorem then guarantees that there exists an essentially unique function  $h$  such that

$$\nu_g(B) = \int_B h(y) dP^Y(y)$$

The class of functions  $h$  is called the conditional expectation of  $g(X)$  given  $Y$  and is denoted by  $h = E[g(X)|Y]$

## Conditional expectation (w.r.t. a r.v.)

Let  $X$  be an integrable r.v. on  $(\Omega, \mathcal{A}, P)$ .

Then we define quite generally  $\mathbb{E}[X|Y]$  as the r.v. such that

(i)  $\mathbb{E}[X|Y]$  is  $\sigma(Y)$ -measurable.

(ii)  $\int_A \mathbb{E}[X|Y](\omega) dP(\omega) = \int_A X(\omega) dP(\omega)$ , for all  $A \in \sigma(Y)$ .

Existence is guaranteed by *Theorem of Radon-Nikodym*.

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In practice:

- ▶ If  $(X, Y)$  is discrete,  $\mathbb{E}[X|Y = y] = \sum_i x_i \mathbb{P}[X = x_i | Y = y]$ .
- ▶ If  $(X, Y)$  is absolutely continuous,

$$\mathbb{E}[X|Y = y] = \frac{1}{f^Y(y)} \int_{\mathbb{R}} x f^{(X,Y)}(x, y) dx = \int_{\mathbb{R}} x f^{(X|Y)}(x, y) dx,$$

where  $f^{(X|Y)}(x, y)$  denotes the pdf of  $X$  conditionally on  $Y = y$ .

## Conditional expectation (w.r.t. a $\sigma$ -algebra)

Let  $X$  be an integrable r.v. on  $(\Omega, \mathcal{A}, \mathbb{P})$ .

Let  $\mathcal{F}$  be a  $\sigma$ -algebra  $\subset \mathcal{A}$ .

Then we define  $\mathbb{E}[X|\mathcal{F}]$  as the r.v. such that

(i)  $\mathbb{E}[X|\mathcal{F}]$  is  $\mathcal{F}$ -measurable.

(ii)  $\int_A \mathbb{E}[X|\mathcal{F}](\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega)$ , for all  $A \in \mathcal{F}$ .

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Same kind of properties as for  $\mathbb{E}[X|Y]$ :

- ▶  $\mathbb{E}[X|\mathcal{F}]$  is  $\mathcal{F}$ -measurable
- ▶  $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$  iff  $X \perp\!\!\!\perp \mathcal{F}$ .
- ▶  $\mathbb{E}[X|\mathcal{F}] = X$  iff  $X$  is  $\mathcal{F}$ -measurable.
- ▶  $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$ .

## Conditional expectation (w.r.t. a $\sigma$ -algebra)

Let  $\mathcal{F}_1 \subset \mathcal{F}_2$  be two  $\sigma$ -algebras in  $\mathcal{A}$ .

Extra properties:

- ▶  $\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1\right] = \mathbb{E}[X|\mathcal{F}_1]$ .
  - ▶  $\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_2\right] = \mathbb{E}[X|\mathcal{F}_1]$ .
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Conditional expectation w.r.t. to a  $\sigma$ -algebra also allows for defining conditional expectation w.r.t. a collection of r.v.'s.

More specifically, we define

$$\mathbb{E}[X|Y_1, \dots, Y_n] := \mathbb{E}[X|\sigma(Y_1, \dots, Y_n)],$$

where  $\sigma(Y_1, \dots, Y_n)$  is the smallest  $\sigma$ -algebra containing  $\{Y_i^{-1}(B) \mid B \in \mathcal{B}, i = 1, \dots, n\}$ .

## Definitions and basic comments

Let  $(\Omega, \mathcal{A}, P)$  be a measure space.

Definition: a filtration is a sequence  $\{\mathcal{A}_n | n \in \mathbb{N}\}$  of  $\sigma$ -algebras such that  $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}$ .

Definition: The SP  $\{Y_n | n \in \mathbb{N}\}$  is adapted to the filtration  $\{\mathcal{A}_n | n \in \mathbb{N}\} \Leftrightarrow Y_n$  is  $\mathcal{A}_n$ -measurable for all  $n$ .

Intuition: growing information  $\mathcal{A}_n \dots$  And the value of  $Y_n$  is known as soon as the information  $\mathcal{A}_n$  is available.

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The SP  $\{Y_n | n \in \mathbb{N}\}$  is a martingale w.r.t. the filtration  $\{\mathcal{A}_n | n \in \mathbb{N}\} \Leftrightarrow$

- ▶ (i)  $\{Y_n | n \in \mathbb{N}\}$  is adapted to the filtration  $\{\mathcal{A}_n | n \in \mathbb{N}\}$ .
- ▶ (ii)  $E[|Y_n|] < \infty$  for all  $n$ .
- ▶ (iii)  $E[Y_{n+1} | \mathcal{A}_n] = Y_n$  a.s. for all  $n$ .

## Definitions and basic comments

Remarks:

- ▶ (iii) shows that a martingale can be thought of as the fortune of a gambler betting on a fair game.
- ▶ (iii)  $\Rightarrow E[Y_n] = E[Y_0]$  for all  $n$  (mean-stationarity). Using (iii), we also have (for  $k = 2, 3, \dots$ )

$$\begin{aligned} E[Y_{n+k} | \mathcal{A}_n] &= E\left[ E[Y_{n+k} | \mathcal{A}_{n+k-1}] | \mathcal{A}_n \right] = E[Y_{n+k-1} | \mathcal{A}_n] = \\ &= \dots = E[Y_{n+1} | \mathcal{A}_n] = Y_n \end{aligned}$$

a.s. for all  $n$ .



## Definitions and basic comments

Let  $\sigma(X_1, \dots, X_n)$  be the smallest  $\sigma$ -algebra containing

$$\{X_i^{-1}(B) \mid B \in \mathcal{B}, i = 1, \dots, n\}.$$

The SP  $\{Y_n \mid n \in \mathbb{N}\}$  is a martingale w.r.t. the SP  $\{X_n \mid n \in \mathbb{N}\} \Leftrightarrow \{Y_n \mid n \in \mathbb{N}\}$  is a martingale w.r.t. the filtration  $\{\sigma(X_1, \dots, X_n) \mid n \in \mathbb{N}\} \Leftrightarrow$

- ▶ (i)  $Y_n$  is  $\sigma(X_1, \dots, X_n)$ -measurable for all  $n$ .
- ▶ (ii)  $E[|Y_n|] < \infty$  for all  $n$ .
- ▶ (iii)  $E[Y_{n+1} \mid X_1, \dots, X_n] = Y_n$  a.s. for all  $n$ .

Remark:

- ▶ (i) just states that " $Y_n$  is a function of  $X_1, \dots, X_n$  only".

## Definitions and basic comments

A lot of examples...

## Example 1

Let  $X_1, X_2, \dots$  be  $\perp\!\!\!\perp$  integrable r.v.'s, with common mean 0.

Let  $Y_n := \sum_{i=1}^n X_i$ .

Then  $\{Y_n\}$  is a martingale w.r.t.  $\{\mathcal{X}_n\}$ .

Indeed,

- ▶ (i) is trivial.
- ▶ (ii) is trivial.
- ▶ (iii): with  $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$ , we have

$$\begin{aligned} \mathbb{E}[Y_{n+1} | \mathcal{A}_n] &= \mathbb{E}[Y_n + X_{n+1} | \mathcal{A}_n] = \mathbb{E}[Y_n | \mathcal{A}_n] + \mathbb{E}[X_{n+1} | \mathcal{A}_n] \\ &= Y_n + \mathbb{E}[X_{n+1}] = Y_n + 0 = Y_n \text{ a.s. for all } n, \end{aligned}$$

where we used that  $Y_n$  is  $\mathcal{A}_n$ -measurable and that  $X_{n+1} \perp\!\!\!\perp \mathcal{A}_n$ .

Similarly, if  $X_1, X_2, \dots$  are  $\perp\!\!\!\perp$  and integrable with means  $\mu_1, \mu_2, \dots$ , respectively,  $\{\sum_{i=1}^n (X_i - \mu_i)\}$  is a martingale w.r.t.  $\{\mathcal{X}_n\}$ .

## Example 2

Let  $X_1, X_2, \dots$  be  $\perp\!\!\!\perp$  integrable r.v.'s, with common mean 1.

Let  $Y_n := \prod_{i=1}^n X_i$ .

Then  $\{Y_n\}$  is a martingale w.r.t.  $\{\mathcal{X}_n\}$ .

Indeed,

- ▶ (i) is trivial.
- ▶ (ii) is trivial.
- ▶ (iii): with  $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$ , we have

$$E[Y_{n+1} | \mathcal{A}_n] = E[Y_n X_{n+1} | \mathcal{A}_n] = Y_n E[X_{n+1} | \mathcal{A}_n] = Y_n E[X_{n+1}] = Y_n$$

a.s. for all  $n$ , where we used that  $Y_n$  is  $\mathcal{A}_n$ -measurable (and hence behaves as a constant in  $E[\cdot | \mathcal{A}_n]$ ).

Similarly, if  $X_1, X_2, \dots$  are  $\perp\!\!\!\perp$  and integrable with means  $\mu_1, \mu_2, \dots$ , respectively,  $\{\prod_{i=1}^n (X_i / \mu_i)\}$  is a martingale w.r.t.  $\{\mathcal{X}_n\}$ .

## Example 2

The previous example is related to basic models for stock prices.

Assume  $X_1, X_2, \dots$  are positive,  $\perp\!\!\!\perp$ , and integrable r.v.'s.

Let  $Y_n := c \prod_{i=1}^n X_i$ , where  $c$  is the initial price.

The quantity  $X_i - 1$  is the change in the value of the stock over a fixed time interval (one day, say) as a fraction of its current value. This multiplicative model (i) ensures nonnegativeness of the price and (ii) is compatible with the fact fluctuations in the value of a stock are roughly proportional to its price.

Various models are obtained for various distributions of the  $X_i$ 's:

- ▶ Discrete Black-Scholes model:  $X_i = e^{\eta_i}$ , where  $\eta_i \sim \mathcal{N}(\mu, \sigma^2)$  for all  $i$ .
- ▶ Binomial model:  $X_i = e^{-r}(1 + a)^{2\eta_i - 1}$ , where  $\eta_i \sim \text{Bin}(1, p)$  for all  $i$  ( $r$  = interest rate, by which one discounts future rewards).

### Example 3: random walks

Let  $X_1, X_2, \dots$  be i.i.d., with  $P[X_i = 1] = p$  and  $P[X_i = -1] = q = 1 - p$

Let  $Y_n := \sum_{i=1}^n X_i$ .

$\leadsto$  The SP  $\{Y_n\}$  is called a random walk.

Remarks:

- ▶ If  $p = q$ , the RW is said to be symmetric.
- ▶ Of course, from Example 1, we know that  $\{(\sum_{i=1}^n X_i) - n(p - q)\}$  is a martingale w.r.t.  $\{X_n\}$ .

But other martingales exist for RWs...

### Example 3: random walks

Consider the non-symmetric case ( $p \neq q$ ) and let  $S_n := \left(\frac{q}{p}\right)^{Y_n}$ .

Then  $\{S_n\}$  is a martingale w.r.t.  $\{X_n\}$ .

Indeed,

- ▶ (i) is trivial.
- ▶ (ii):  $|S_n| \leq \max((q/p)^n, (q/p)^{-n})$ . Hence,  $E[|S_n|] < \infty$ .
- ▶ (iii): with  $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$ , we have

$$\begin{aligned} E[S_{n+1} | \mathcal{A}_n] &= E[(q/p)^{Y_n} (q/p)^{X_{n+1}} | \mathcal{A}_n] = (q/p)^{Y_n} E[(q/p)^{X_{n+1}} | \mathcal{A}_n] \\ &= S_n E[(q/p)^{X_{n+1}}] = S_n \left( (q/p)^1 \times p + (q/p)^{-1} \times q \right) = S_n, \text{ a.s. for all } n \end{aligned}$$

where we used that  $Y_n$  is  $\mathcal{A}_n$ -measurable and that  $X_{n+1} \perp\!\!\!\perp \mathcal{A}_n$ .

### Example 3: random walks

Consider the symmetric case ( $p = q$ ) and let  $S_n := Y_n^2 - n$ .

Then  $\{S_n\}$  is a martingale w.r.t.  $\{X_n\}$ .

Indeed,

- ▶ (i) is trivial.
- ▶ (ii):  $|S_n| \leq n^2 - n$ . Hence,  $E[|S_n|] < \infty$ .
- ▶ (iii): with  $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$ , we have

$$\begin{aligned} E[S_{n+1} | \mathcal{A}_n] &= E[(Y_n + X_{n+1})^2 - (n+1) | \mathcal{A}_n] \\ &= E[(Y_n^2 + X_{n+1}^2 + 2Y_n X_{n+1}) - (n+1) | \mathcal{A}_n] \\ &= Y_n^2 + E[X_{n+1}^2 | \mathcal{A}_n] + 2Y_n E[X_{n+1} | \mathcal{A}_n] - (n+1) \\ &= Y_n^2 + E[X_{n+1}^2] + 2Y_n E[X_{n+1}] - (n+1) \\ &= S_n \text{ a.s. for all } n, \end{aligned}$$

where we used that  $Y_n$  is  $\mathcal{A}_n$ -measurable and that  $X_{n+1} \perp\!\!\!\perp \mathcal{A}_n$ .



## Example 4: De Moivre's martingales

Let  $X_1, X_2, \dots$  be i.i.d., with  $P[X_i = 1] = p$  and  $P[X_i = -1] = q = 1 - p$ .

Let  $Y_0 := k \in \{1, 2, \dots, m-1\}$  be the initial state.

Let  $Y_{n+1} := (Y_n + X_{n+1}) I[Y_n \notin \{0, m\}] + Y_n I[Y_n \in \{0, m\}]$ .

$\leadsto$  The SP  $\{Y_n\}$  is called a random walk with absorbing barriers.

Remarks:

- ▶ Before being caught either in 0 or  $m$ , this is just a RW.
- ▶ As soon as you get in 0 or  $m$ , you stay there forever.

## Example 4: De Moivre's martingales

Let  $X_1, X_2, \dots$  be i.i.d., with  $P[X_i = 1] = p$  and  $P[X_i = -1] = q = 1 - p$

Let  $Y_0 := k \in \{1, 2, \dots, m-1\}$  be the initial state.

Let  $Y_{n+1} := (Y_n + X_{n+1}) I[Y_n \notin \{0, m\}] + Y_n I[Y_n \in \{0, m\}]$ .

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In this new setup and with this new definition of  $Y_n$ ,

- ▶ In the non-symmetric case,

$\{S_n := \left(\frac{q}{p}\right)^{Y_n}\}$  is still a martingale w.r.t.  $\{X_n\}$ .

(exercise).

## Example 5: branching processes

Consider some population, in which each individual  $i$  of the  $Z_n$  individuals in the  $n$ th generation gives birth to  $X_{n+1,i}$  children (the  $X_{n,i}$ 's are i.i.d., take values in  $\mathbb{N}$ , and have common mean  $\mu < \infty$ ).

Assume that  $Z_0 = 1$ .

Then  $\{Z_n/\mu^n\}$  is a martingale w.r.t.  $\{\mathcal{A}_n := \sigma(\text{the } X_{m,i}\text{'s, } m \leq n)\}$ .

Indeed,

- ▶ (i), (ii): exercise...
- ▶ (iii): 
$$\begin{aligned} \mathbb{E}\left[\frac{Z_{n+1}}{\mu^{n+1}} \mid \mathcal{A}_n\right] &= \frac{1}{\mu^{n+1}} \mathbb{E}\left[\sum_{i=1}^{Z_n} X_{n+1,i} \mid \mathcal{A}_n\right] \\ &= \frac{1}{\mu^{n+1}} \sum_{i=1}^{Z_n} \mathbb{E}[X_{n+1,i} \mid \mathcal{A}_n] = \frac{1}{\mu^{n+1}} \sum_{i=1}^{Z_n} \mathbb{E}[X_{n+1,i}] = \frac{Z_n}{\mu^n} \text{ a.s. for all } n. \end{aligned}$$

In particular,  $\mathbb{E}\left[\frac{Z_n}{\mu^n}\right] = \mathbb{E}\left[\frac{Z_0}{\mu^0}\right] = 1$ . Hence,  $\mathbb{E}[Z_n] = \mu^n$  for all  $n$ .

## Example 6: Polya's urn

Consider an urn containing  $b$  blue balls and  $r$  red ones. Pick randomly some ball in the urn and put it back in the urn with an extra ball of the same color. Repeat this procedure.

This is a so-called contamination process.

Let  $X_n$  be the number of red balls in the urn after  $n$  steps.

Let

$$R_n = \frac{X_n}{b + r + n}$$

be the proportion of red balls in the urn after  $n$  steps.

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Then  $\{R_n\}$  is a martingale w.r.t.  $\{X_n\}$ .

## Example 6: Polya's urn

Indeed,

- ▶ (i) is trivial.
- ▶ (ii):  $0 \leq |R_n| \leq 1$ . Hence,  $E[|R_n|] < \infty$ .
- ▶ (iii): with  $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$ , we have

$$\begin{aligned} E[X_{n+1} | \mathcal{A}_n] &= (X_n + 1) \frac{X_n}{r + b + n} + (X_n + 0) \left( 1 - \frac{X_n}{r + b + n} \right) \\ &= \frac{(X_n + 1)X_n + X_n((r + b + n) - X_n)}{r + b + n} \\ &= \frac{(r + b + n + 1)X_n}{r + b + n} \\ &= (r + b + n + 1)R_n \text{ a.s. for all } n, \end{aligned}$$

so that  $E[R_{n+1} | \mathcal{A}_n] = R_n$  a.s. for all  $n$ .

## Stopping times

Let  $(Y_n)$  be a martingale w.r.t.  $(\mathcal{A}_n)$

Let  $T : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$  be a r.v.

Definition:  $T$  is a **stopping time w.r.t.  $(\mathcal{A}_n)$**   $\Leftrightarrow$

(i)  $T$  is **a.s. finite** (i.e.,  $\mathbb{P}[T < \infty] = 1$ ).

(ii)  $[T = n] \in \mathcal{A}_n$  for all  $n$ .

### Remarks:

- ▶ (ii) is the crucial assumption:  
it says that one knows, at time  $n$ , on the basis of the "information"  $\mathcal{A}_n$ , whether  $T = n$  or not, that is, whether one should stop at  $n$  or not.
- ▶ (i) just makes (almost) sure that one will stop at some point.

## Stopping times (examples)

(Kind of) examples...

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Let  $(Y_n)$  be a martingale w.r.t.  $(\mathcal{A}_n)$ . Let  $B \in \mathcal{B}$ .

(A) Let  $T := \inf\{n \in \mathbb{N} \mid Y_n \in B\}$  be the time of 1st entry of  $(Y_n)$  into  $B$ . Then,

$$[T = n] = [Y_0 \notin B, Y_1 \notin B, \dots, Y_{n-1} \notin B, Y_n \in B] \in \mathcal{A}_n.$$

Hence, provided that  $T$  is a.s. finite,  $T$  is a ST.

(B) Let  $T := \sup\{n \in \mathbb{N} \mid Y_n \in B\}$  be the time of last escape of  $(Y_n)$  out of  $B$ . Then,

$$[T = n] = [Y_n \in B, Y_{n+1} \notin B, Y_{n+2} \notin B, \dots] \notin \mathcal{A}_n.$$

Hence,  $T$  is not a ST.

## Stopping times (examples)

(C) Let  $T := k$  a.s. (for some fixed integer  $k$ ). Then, of course,  
(i)  $T < \infty$  a.s. and (ii)

$$[T = n] = \begin{cases} \emptyset & \text{if } n \neq k \\ \Omega & \text{if } n = k, \end{cases}$$

which is in  $\mathcal{A}_n$  for all  $n$ . Hence,  $T$  is a ST.



## Stopping times (properties)

Properties:

- ▶  $[T = n] \in \mathcal{A}_n \forall n \stackrel{(1)}{\Leftrightarrow} [T \leq n] \in \mathcal{A}_n \forall n \stackrel{(2)}{\Leftrightarrow} [T > n] \in \mathcal{A}_n \forall n.$

Indeed,

$\stackrel{(1)}{\Rightarrow}$  follows from  $[T \leq n] = \bigcup_{k=1}^n [T = k].$

$\stackrel{(1)}{\Leftarrow}$  follows from  $[T = n] = [T \leq n] \setminus [T \leq n - 1].$

$\stackrel{(2)}{\Leftrightarrow}$  follows from  $[T \leq n] = \Omega \setminus [T > n].$

- ▶  $T_1, T_2$  are ST  $\Rightarrow T_1 + T_2, \max(T_1, T_2),$  and  $\min(T_1, T_2)$  are ST (exercise).
- ▶ Let  $(Y_n)$  be a martingale w.r.t.  $(\mathcal{A}_n).$   
Let  $T$  be a ST w.r.t.  $(\mathcal{A}_n).$   
Then  $Y_T := \sum_{n=0}^{\infty} Y_n \mathbb{I}_{[T=n]}$  is a r.v.  
Indeed,  $[Y_T \in B] = \bigcup_{n=0}^{\infty} \{[T = n] \cap [Y_n \in B]\} \in \mathcal{A}.$

## Stopped martingale

A key lemma:

**Lemma:** Let  $(Y_n)$  be a martingale w.r.t.  $(\mathcal{A}_n)$ . Let  $T$  be a ST w.r.t.  $(\mathcal{A}_n)$ .

Then  $\{Z_n := Y_{\min(n,T)}\}$  is a martingale w.r.t.  $(\mathcal{A}_n)$ .

Proof: note that

$$Z_n = Y_{\min(n,T)} = \sum_{k=0}^{n-1} Y_k \mathbb{I}_{[T=k]} + Y_n \mathbb{I}_{[T \geq n]}.$$

So

- ▶ (i):  $Z_n$  is  $\mathcal{A}_n$ -measurable for all  $n$ .
- ▶ (ii):  $|Z_n| \leq \sum_{k=0}^{n-1} |Y_k| \mathbb{I}_{[T=k]} + |Y_n| \mathbb{I}_{[T \geq n]} \leq \sum_{k=0}^n |Y_k|$ .  
Hence,  $\mathbb{E}[|Z_n|] < \infty$ .

## Stopped martingale

- (iii): we have

$$\begin{aligned}\mathbb{E}[Z_{n+1}|\mathcal{A}_n] - Z_n &= \mathbb{E}[Z_{n+1} - Z_n|\mathcal{A}_n] \\ &= \mathbb{E}[(Y_{n+1} - Y_n)\mathbb{I}_{[T \geq n+1]}|\mathcal{A}_n] \\ &= \mathbb{E}[(Y_{n+1} - Y_n)\mathbb{I}_{[T > n]}|\mathcal{A}_n] \\ &= \mathbb{I}_{[T > n]} \mathbb{E}[Y_{n+1} - Y_n|\mathcal{A}_n] \\ &= \mathbb{I}_{[T > n]} (\mathbb{E}[Y_{n+1}|\mathcal{A}_n] - Y_n) \\ &= \mathbf{0} \text{ a.s. for all } n,\end{aligned}$$

where we used that  $\mathbb{I}_{[T > n]}$  is  $\mathcal{A}_n$ -measurable. □

## Stopped martingale

**Corollary:** Let  $(Y_n)$  be a martingale w.r.t.  $(\mathcal{A}_n)$ . Let  $T$  be a ST w.r.t.  $(\mathcal{A}_n)$ . Then  $\mathbb{E}[Y_{\min(n,T)}] = \mathbb{E}[Y_0]$  for all  $n$ .

Proof: the lemma and the mean-stationarity of martingales yield  $\mathbb{E}[Y_{\min(n,T)}] = \mathbb{E}[Z_n] = \mathbb{E}[Z_0] = \mathbb{E}[Y_{\min(0,T)}] = \mathbb{E}[Y_0]$  for all  $n$ .  $\square$

---

In particular, if the ST is such that  $T \leq k$  a.s. for some  $k$ , we have that, for  $n \geq k$ ,

$$Y_{\min(n,T)} = Y_T \text{ a.s.,}$$

so that

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0].$$

## Stopped martingale

$\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$  does not always hold.

---

Example: the doubling strategy, for which the winnings are

$$Y_n = \sum_{i=1}^n C_i X_i,$$

where the  $X_i$ 's are i.i.d.  $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}$  and  $C_i = 2^{i-1} b$ .

The SP  $(Y_n)$  is a martingale w.r.t.  $(X_n)$  (exercise).

Let  $T = \inf\{n \in \mathbb{N} | X_n = 1\}$  (exercise:  $T$  is a ST).

As we have seen,  $Y_T = b$  a.s., so that  $\mathbb{E}[Y_T] = b \neq 0 = \mathbb{E}[Y_0]$ .

---

However, as shown by the following result,  $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$  holds under much broader conditions than " $T \leq k$  a.s."

## Optional stopping theorem

**Theorem:** Let  $(Y_n)$  be a martingale w.r.t.  $(\mathcal{A}_n)$ . Let  $T$  be a ST w.r.t.  $(\mathcal{A}_n)$ . Then if (i)  $\mathbb{E}[|Y_T|] < \infty$  and (ii)  $\lim_{n \rightarrow \infty} \mathbb{E}[Y_n \mathbb{I}_{[T > n]}] = 0$ , we have  $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$ .

Proof: since  $Y_{\min(n, T)} = Y_n \mathbb{I}_{[T > n]} + Y_T \mathbb{I}_{[T \leq n]}$ , we have

$$Y_T = Y_T \mathbb{I}_{[T \leq n]} + Y_T \mathbb{I}_{[T > n]} = (Y_{\min(n, T)} - Y_n \mathbb{I}_{[T > n]}) + Y_T \mathbb{I}_{[T > n]}.$$

Taking expectations, we obtain

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0] - \mathbb{E}[Y_n \mathbb{I}_{[T > n]}] + \mathbb{E}[Y_T \mathbb{I}_{[T > n]}].$$

By taking the limit as  $n \rightarrow \infty$  and using (ii),

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0] + \lim_{n \rightarrow \infty} \mathbb{E}[Y_T \mathbb{I}_{[T > n]}].$$

The result follows from  $\lim_{n \rightarrow \infty} \mathbb{P}[T > n] = \mathbb{P}[T = \infty] = 0$ .  $\square$

## Optional stopping theorem

**Theorem:** Let  $(Y_n)$  be a martingale w.r.t.  $(\mathcal{A}_n)$ . Let  $T$  be a ST w.r.t.  $(\mathcal{A}_n)$ . Then if (i)  $\mathbb{E}[|Y_T|] < \infty$  and (ii)  $\lim_{n \rightarrow \infty} \mathbb{E}[Y_n \mathbb{I}_{\{T > n\}}] = 0$ , we have  $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$ .

Particular sufficient conditions for (i), (ii):

- ▶ (a)  $T \leq k$  a.s. Indeed,

(i)  $\mathbb{E}[|Y_T|] = \mathbb{E}[|\sum_{n=0}^k Y_n \mathbb{I}_{\{T \geq n\}}|] \leq \sum_{n=0}^k \mathbb{E}[|Y_n|] < \infty.$

(ii)  $Y_n \mathbb{I}_{\{T > n\}} = 0$  a.s. for  $n > k$ . Hence,  $\mathbb{E}[Y_n \mathbb{I}_{\{T > n\}}] = 0$  for  $n > k$ , so that (ii) holds.

- ▶ (b)  $(Y_n)$  is uniformly integrable.

## Optional stopping theorem (examples)

Let  $p_k := \mathbb{P}[Y_T = 0]$ .

Then

$$\mathbb{E}[Y_T] = 0 \times p_k + m \times (1 - p_k)$$

and

$$\mathbb{E}[Y_0] = \mathbb{E}[k] = k,$$

so that  $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$  yields

$$m(1 - p_k) = k,$$

that is, solving for  $p_k$ ,

$$p_k = \frac{m - k}{m}.$$



## Optional stopping theorem (examples)

Let  $X_1, X_2, \dots$  be i.i.d., with  $\mathbb{P}[X_i = 1] = p$  and  $\mathbb{P}[X_i = -1] = q = 1 - p$ .

Let  $Y_0 := k \in \{1, 2, \dots, m-1\}$  be the initial state.

Let  $Y_{n+1} := (Y_n + X_{n+1}) \mathbb{I}_{[Y_n \notin \{0, m\}]} + Y_n \mathbb{I}_{[Y_n \in \{0, m\}]}$ .

$\rightsquigarrow$  The SP  $(Y_n)$  is called a random walk with absorbing barriers.

In the non-symmetric case,  $\left(S_n := \left(\frac{q}{p}\right)^{Y_n}\right)$  is a martingale w.r.t.  $(X_n)$ . Let  $T := \inf\{n \in \mathbb{N} \mid S_n \in \{0, m\}\}$  (exercise:  $T$  is a stopping time, and the assumptions of the optional stopping thm are satisfied).  $\rightsquigarrow \mathbb{E}[S_T] = \mathbb{E}[S_0]$ .

## Optional stopping theorem (examples)

Let again  $p_k := \mathbb{P}[Y_T = 0]$ . Then, since

$$\mathbb{E}[S_T] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{Y_T}\right] = \left(\frac{q}{p}\right)^0 \times p_k + \left(\frac{q}{p}\right)^m (1 - p_k),$$

and

$$\mathbb{E}[S_0] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{Y_0}\right] = \mathbb{E}\left[\left(\frac{q}{p}\right)^k\right] = \left(\frac{q}{p}\right)^k,$$

we deduce that

$$\left(\frac{q}{p}\right)^0 \times p_k + \left(\frac{q}{p}\right)^m (1 - p_k) = \left(\frac{q}{p}\right)^k,$$

that is, solving for  $p_k$ ,

$$p_k = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^m}.$$

## Optional stopping theorem (examples)

Is there a way to get  $\mathbb{E}[T]$  here as well?

In the non-symmetric case,  $(R_n := Y_n - \min(n, T)(p - q))$  is a martingale w.r.t.  $(X_n)$  (exercise: check this, and check that the optional stopping thm applies with  $(R_n)$  and  $T$ ).

$\leadsto \mathbb{E}[R_T] = \mathbb{E}[R_0]$ , where

$$\mathbb{E}[R_T] = \mathbb{E}[Y_T - T(p - q)] = \mathbb{E}[Y_T] - (p - q)\mathbb{E}[T] = \left(0 \times p_k + m \times (1 - p_k)\right)$$

and

$$\mathbb{E}[R_0] = \mathbb{E}[Y_0 - \min(0, T)(p - q)] = \mathbb{E}[Y_0] = \mathbb{E}[k] = k.$$

Hence,

$$\mathbb{E}[T] = \frac{m(1 - p_k) - k}{p - q} = \frac{m\left(1 - \left(\frac{q}{p}\right)^k\right) - k\left(1 - \left(\frac{q}{p}\right)^m\right)}{(p - q)\left(1 - \left(\frac{q}{p}\right)^m\right)}.$$

## Sub- and super-martingales

Not every game is fair...

There are also favourable and defavourable games.

---

Therefore, we introduce the following concepts:

The SP  $(Y_n)_{n \in \mathbb{N}}$  is a submartingale w.r.t. the filtration  $(\mathcal{A}_n)_{n \in \mathbb{N}} \Leftrightarrow$

- ▶ (i)  $(Y_n)_{n \in \mathbb{N}}$  is adapted to the filtration  $(\mathcal{A}_n)_{n \in \mathbb{N}}$ .
- ▶ (ii)'  $\mathbb{E}[Y_n^+] < \infty$  for all  $n$ .
- ▶ (iii)'  $\mathbb{E}[Y_{n+1} | \mathcal{A}_n] \geq Y_n$  a.s. for all  $n$ .

The SP  $(Y_n)_{n \in \mathbb{N}}$  is a supermartingale w.r.t. the filtration  $(\mathcal{A}_n)_{n \in \mathbb{N}} \Leftrightarrow$

- ▶ (i)  $(Y_n)_{n \in \mathbb{N}}$  is adapted to the filtration  $(\mathcal{A}_n)_{n \in \mathbb{N}}$ .
- ▶ (ii)''  $\mathbb{E}[Y_n^-] < \infty$  for all  $n$ .
- ▶ (iii)''  $\mathbb{E}[Y_{n+1} | \mathcal{A}_n] \leq Y_n$  a.s. for all  $n$ .

## Sub- and super-martingales

Remarks:

- ▶ (iii)' shows that a submartingale can be thought of as the fortune of a gambler betting on a favourable game.
- ▶ (iii)'  $\Rightarrow \mathbb{E}[Y_n] \geq \mathbb{E}[Y_0]$  for all  $n$ .
- ▶ (iii)'' shows that a supermartingale can be thought of as the fortune of a gambler betting on a defavourable game.
- ▶ (iii)''  $\Rightarrow \mathbb{E}[Y_n] \leq \mathbb{E}[Y_0]$  for all  $n$ .
- ▶  $(Y_n)$  is a submartingale w.r.t.  $(\mathcal{A}_n)$   
 $\Leftrightarrow (-Y_n)$  is a supermartingale w.r.t.  $(\mathcal{A}_n)$ .
- ▶  $(Y_n)$  is a martingale w.r.t.  $(\mathcal{A}_n)$   
 $\Leftrightarrow (Y_n)$  is both a sub- and a supermartingale w.r.t.  $(\mathcal{A}_n)$ .

## Sub- and super-martingales

Consider the following strategy for the fair version of roulette (without the "0" slot):

Bet  $b$  euros on an even result. If you win, stop.

If you lose, bet  $2b$  euros on an even result. If you win, stop.

If you lose, bet  $4b$  euros on an even result. If you win, stop...

And so on...

How good is this strategy?

(a) If you first win in the  $n$ th game, your total winning is

$$- \sum_{i=0}^{n-2} 2^i b + 2^{n-1} b = b$$

⇒ Whatever the value of  $n$  is, you win  $b$  euros with this strategy.

## Sub- and super-martingales

(b) You will a.s. win. Indeed, let  $T$  be the time index of first success. Then

$$\begin{aligned}\mathbb{P}[T < \infty] &= \sum_{n=1}^{\infty} \mathbb{P}[n-1 \text{ first results are "odd", then "even"}] \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1.\end{aligned}$$

But

(c) The expected amount you lose just before you win is

$$0 \times \frac{1}{2} + b \times \left(\frac{1}{2}\right)^2 + (b+2b) \times \left(\frac{1}{2}\right)^3 + \dots + \left(\sum_{i=0}^{n-2} 2^i b\right) \left(\frac{1}{2}\right)^n + \dots = \infty$$

$\Rightarrow$  Your expected loss is infinite!

(d) You need an unbounded wallet...

## Sub- and super-martingales

Let us try to formalize strategies...

---

Consider the SP  $(X_n)_{n \in \mathbb{N}}$ , where  $X_n$  is your winning per unit stake in game  $n$ . Denote by  $(\mathcal{A}_n)_{n \in \mathbb{N}}$  the corresponding filtration ( $\mathcal{A}_n = \sigma(X_1, \dots, X_n)$ ).

Definition: A gambling strategy (w.r.t.  $(X_n)$ ) is a SP  $(C_n)_{n \in \mathbb{N}}$  such that  $C_n$  is  $\mathcal{A}_{n-1}$ -measurable for all  $n$ .

Remarks:

- ▶  $C_n = C_n(X_1, \dots, X_{n-1})$  is what you will bet in game  $n$ .
- ▶  $\mathcal{A}_0 = \{\emptyset, \Omega\}$ .



## Sub- and super-martingales

Using some strategy  $(C_n)$ , your total winning after  $n$  games is

$$Y_n^{(C)} = \sum_{i=1}^n C_i X_i.$$

A natural question:

Is there any way to choose  $(C_n)$  so that  $(Y_n^{(C)})$  is "nice"?

Consider the "blind" strategy  $C_n = 1$  (for all  $n$ ), that consists in betting 1 euro in each game, and denote by  $(Y_n = \sum_{i=1}^n X_i)$  the corresponding process of winnings.

Then, here is the answer:

**Theorem:** Let  $(C_n)$  be a gambling strategy with nonnegative and bounded r.v.'s. Then if  $(Y_n)$  is a martingale, so is  $(Y_n^{(C)})$ . If  $(Y_n)$  is a submart., so is  $(Y_n^{(C)})$ . And if  $(Y_n)$  is a supermart., so is  $(Y_n^{(C)})$ .

## Sub- and super-martingales

Proof:

- ▶ (i) is trivial.
- ▶ (ii):  $|Y_n^{(C)}| \leq \sum_{i=1}^n a_i |X_i|$ . Hence,  $\mathbb{E}[|Y_n^{(C)}|] < \infty$ .
- ▶ (iii), (iii)', (iii)”: with  $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$ , we have

$$\begin{aligned}\mathbb{E}[Y_{n+1}^{(C)} | \mathcal{A}_n] &= \mathbb{E}[Y_n^{(C)} + C_{n+1} X_{n+1} | \mathcal{A}_n] \\ &= Y_n^{(C)} + C_{n+1} \mathbb{E}[X_{n+1} | \mathcal{A}_n] \\ &= Y_n^{(C)} + C_{n+1} \mathbb{E}[Y_{n+1} - Y_n | \mathcal{A}_n] \\ &= Y_n^{(C)} + C_{n+1} (\mathbb{E}[Y_{n+1} | \mathcal{A}_n] - Y_n),\end{aligned}$$

where we used that  $C_{n+1}$  is  $\mathcal{A}_n$ -measurable. Since  $C_{n+1} \geq 0$ , the result follows. □

Remark: The second part was checked for martingales only.  
Exercise: check (ii)' and (ii)”...

## Convergence of martingales

**Theorem:** let  $(Y_n)$  be a submartingale w.r.t.  $(\mathcal{A}_n)$ . Assume that, for some  $M$ ,  $\mathbb{E}[Y_n^+] \leq M$  for all  $n$ . Then

- (i)  $\exists Y_\infty$  such that  $Y_n \xrightarrow{a.s.} Y_\infty$  as  $n \rightarrow \infty$ .
  - (ii) If  $\mathbb{E}[|Y_0|] < \infty$ ,  $\mathbb{E}[|Y_\infty|] < \infty$ .
- 

The following results directly follow:

**Corollary 1:** let  $(Y_n)$  be a submartingale or a supermartingale w.r.t.  $(\mathcal{A}_n)$ . Assume that, for some  $M$ ,  $\mathbb{E}[|Y_n|] \leq M$  for all  $n$ . Then  $\exists Y_\infty$  (satisfying  $\mathbb{E}[|Y_\infty|] < \infty$ ) such that  $Y_n \xrightarrow{a.s.} Y_\infty$  as  $n \rightarrow \infty$ .

**Corollary 2:** let  $(Y_n)$  be a negative submartingale or a positive supermartingale w.r.t.  $(\mathcal{A}_n)$ . Then  $\exists Y_\infty$  such that  $Y_n \xrightarrow{a.s.} Y_\infty$  as  $n \rightarrow \infty$ .

## Example: products of r.v.'s

Let  $X_1, X_2, \dots$  be i.i.d. r.v.'s, with common distribution

distribution of $X_i$		
values	0	2
probabilities	$\frac{1}{2}$	$\frac{1}{2}$

The  $X_i$ 's are integrable r.v.'s with common mean 1, so that  $(Y_n = \prod_{i=1}^n X_i)$  is a (positive) martingale w.r.t.  $(X_n)$  (example 2 in the previous lecture).

Consequently,  $\exists Y_\infty$  such that  $Y_n \xrightarrow{a.s.} Y_\infty$  as  $n \rightarrow \infty$ .

We showed, in Lecture #4, that  $Y_n \xrightarrow{P} 0$  as  $n \rightarrow \infty$  so that  $Y_\infty = 0$  a.s.

---

But we also showed there that convergence in  $L^1$  does not hold. To ensure  $L^1$ -convergence, one has to require uniform integrability of  $(Y_n)$ .

## Example: Polya's urn

Consider an urn containing  $b$  blue balls and  $r$  red ones. Pick randomly some ball in the urn and put it back in the urn with an extra ball of the same color. Repeat this procedure.

Let  $X_n$  be the number of red balls in the urn after  $n$  steps.

Let  $R_n = \frac{X_n}{b+r+n}$  be the proportion of red balls after  $n$  steps.

We know that  $(R_n)$  is a martingale w.r.t.  $(X_n)$ .

---

Now,  $|R_n| \leq 1$  ( $\Rightarrow \mathbb{E}[|R_n|] \leq 1$ ), so that  $\exists R_\infty$  (satisfying  $\mathbb{E}[|R_\infty|] < \infty$ ) such that  $R_n \xrightarrow{\text{a.s.}} R_\infty$  as  $n \rightarrow \infty$ .

Clearly, uniform integrability holds. Hence,  $R_n \xrightarrow{L^1} R_\infty$  as  $n \rightarrow \infty$ .

Remark: it can be shown that  $R_\infty$  has a beta distribution:

$$\mathbb{P}[R_\infty \leq u] = \binom{b+r}{r} \int_0^u x^{r-1} (1-x)^{b-1} dx, \quad u \in (0, 1).$$

## Example: branching processes

Consider some population, in which each individual  $i$  of the  $Z_n$  individuals in the  $n$ th generation gives birth to  $X_{n,i}$  children (the  $X_{n,i}$ 's are i.i.d., take values in  $\mathbb{N}$ , and have common mean  $\mu < \infty$ ).

Assume that  $Z_0 = 1$ .

Then  $(Y_n := Z_n/\mu^n)$  is a martingale w.r.t.  $(\mathcal{A}_n := \sigma(X_{n,i}, n \text{ fixed}))$ .

---

What can be said about the long-run state?

$\mathbb{E}[|Y_n|] = \mathbb{E}[Y_0] = 1$  for all  $n \Rightarrow \exists Y_\infty$  (satisfying  $\mathbb{E}[|Y_\infty|] < \infty$ ) such that  $Y_n \xrightarrow{\text{a.s.}} Y_\infty$  as  $n \rightarrow \infty$ .

$\leadsto Z_n \approx Y_\infty \mu^n$  for large  $n$ .

## Example: branching processes

Case 1:  $\mu \leq 1$ .

If  $p_0 := \mathbb{P}[X_{n,i} = 0] > 0$ ,  $Z_n \xrightarrow{\text{a.s.}} Z_\infty := 0$  as  $n \rightarrow \infty$ .

Proof:

Let  $k \in \mathbb{N}_0$ . Then

$$\mathbb{P}[Z_\infty = k] = \mathbb{P}[Z_n = k \ \forall n \geq n_0] = \mathbb{P}[Z_{n_0} = k] (\mathbb{P}[Z_{n_0+1} = k | Z_{n_0} = k])^\infty$$

where

$$\mathbb{P}[Z_{n_0+1} = k | Z_{n_0} = k] \leq \mathbb{P}[Z_{n_0+1} \neq 0 | Z_{n_0} = k] = 1 - (p_0)^k < 1.$$

Therefore,  $\mathbb{P}[Z_\infty = k] = 0$  for all  $k \in \mathbb{N}_0$ , so that  $Z_\infty = 0$  a.s.  $\square$

## Example: branching processes

Case 2:  $\mu > 1$ .

Then  $Z_n \xrightarrow{\text{a.s.}} Z_\infty$  as  $n \rightarrow \infty$ , where the distribution of  $Z_\infty$  is given by

distribution of $Z_\infty$		
values	0	$\infty$
probabilities	$\pi$	$1 - \pi$

where  $\pi \in (0, 1)$  is the unique solution of  $g(s) = s$ , where, defining  $p_k := \mathbb{P}[X_{n,i} = k]$ , we let  $g(s) := \sum_{k=0}^{\infty} p_k s^k$ .

Proof:

Clearly, since  $Z_n \approx Y_\infty \mu^n$  for large  $n$ ,  $Z_n \xrightarrow{\text{a.s.}} Z_\infty$  as  $n \rightarrow \infty$ , where  $Z_\infty$  only assumes the values 0 and  $\infty$ , with proba  $a$  and  $1 - a$ , respectively, say.

We may assume that  $p_0 > 0$  (clearly, if  $p_0 = 0$ ,  $Z_\infty = \infty$  a.s.)



## Example: branching processes

There is indeed a unique  $\pi \in (0, 1)$  such that  $g(\pi) = \pi$ , since  $g$  is monotone increasing,  $g(0) = p_0 > 0$ , and  $g'(1) = \mu > 1$ .

$\leadsto (\pi^{Z_n})$  is a martingale w.r.t.  $(\mathcal{A}_n)$ , since

$$\begin{aligned}\mathbb{E}[\pi^{Z_{n+1}} | \mathcal{A}_n] &= \mathbb{E}[\pi^{\sum_{i=1}^{Z_n} X_{n+1,i}} | \mathcal{A}_n] = (\mathbb{E}[\pi^{X_{n+1,1}} | \mathcal{A}_n])^{Z_n} \\ &= (\mathbb{E}[\pi^{X_{n+1,1}}])^{Z_n} = \left( \sum_{k=0}^{\infty} \pi^k p_k \right)^{Z_n} = (g(\pi))^{Z_n} = \pi^{Z_n}\end{aligned}$$

a.s. for all  $n$ .

$(\pi^{Z_n})$  is also uniformly integrable, so that  $\pi^{Z_n} \xrightarrow{L^1} \pi^{Z_\infty}$  as  $n \rightarrow \infty$ . Therefore,  $\mathbb{E}[\pi^{Z_\infty}] = \pi^0 \times a + \pi^\infty \times (1 - a) = a$  is equal to  $\mathbb{E}[\pi^{Z_0}] = \mathbb{E}[\pi^1] = \pi$ .

Hence,  $\mathbb{P}[Z_\infty = 0] = \pi$  and  $\mathbb{P}[Z_\infty = \infty] = 1 - \pi$ . □