Stochastic Processes (Lecture #5)

Thomas Verdebout

Université Libre de Bruxelles

Outline of the course

1. A short introduction.
2. Basic probability review.
3. $\boxed{\text{3. Martingales.}}$
4. Markov chains.
5. Markov processes, Poisson processes.
6. Brownian motions.

Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
   - 3.1. Definitions and examples.
   - 3.2. Sub- and super- martingales, gambling strategies.
   - 3.3. Stopping times and the optional stopping theorem.
   - 3.4. Limiting results.
4. Markov chains.
5. Markov processes, Poisson processes.
6. Brownian motions.

Definitions and basic comments

Let $(\Omega, \mathcal{A}, P)$ be a measure space.

Definition: a filtration is a sequence $\{\mathcal{A}_n | n \in \mathbb{N}\}$ of σ-algebras such that $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}$.

Definition: The SP $\{Y_n | n \in \mathbb{N}\}$ is adapted to the filtration $\{\mathcal{A}_n | n \in \mathbb{N}\}$ $\Leftrightarrow$ Y_n is $\mathcal{A}_n$-measurable for all n.

Intuition: growing information $\mathcal{A}_n$... And the value of Y_n is known as soon as the information $\mathcal{A}_n$ is available.

The SP $\{Y_n | n \in \mathbb{N}\}$ is a martingale w.r.t. the filtration $\{\mathcal{A}_n | n \in \mathbb{N}\}$ $\Leftrightarrow$

- (i) $\{Y_n | n \in \mathbb{N}\}$ is adapted to the filtration $\{\mathcal{A}_n | n \in \mathbb{N}\}$.
- (ii) $\mathrm{E}[|Y_n|] < \infty$ for all n.
- (iii) $\mathrm{E}[Y_{n+1} | \mathcal{A}_n] = Y_n$ a.s. for all n.

Definitions and basic comments

Remarks:

- (iii) shows that a martingale can be thought of as the fortune of a gambler betting on a fair game.
- (iii) $\Rightarrow \mathrm{E}[Y_n] = \mathrm{E}[Y_0]$ for all n (mean-stationarity). Using (iii), we also have (for $k = 2, 3, \dots$)

$$\mathrm{E}[Y_{n+k}|\mathcal{A}_n] = \mathrm{E}\Big[\mathrm{E}[Y_{n+k}|\mathcal{A}_{n+k-1}]\Big|\mathcal{A}_n\Big] = \mathrm{E}[Y_{n+k-1}|\mathcal{A}_n] = \\ = \dots = \mathrm{E}[Y_{n+1}|\mathcal{A}_n] = Y_n$$

a.s. for all n.

Definitions and basic comments

Let $\sigma(X_1, \ldots, X_n)$ be the smallest σ-algebra containing

$$\{X_i^{-1}(B) \,|\, B \in \mathcal{B},\, i = 1, \ldots, n\}.$$

The SP $\{Y_n | n \in \mathbb{N}\}$ is a martingale w.r.t. the SP $\{X_n | n \in \mathbb{N}\} \Leftrightarrow$
$\{Y_n | n \in \mathbb{N}\}$ is a martingale w.r.t. the filtration $\{\sigma(X_1, \ldots, X_n) | n \in \mathbb{N}\} \Leftrightarrow$

- (i) Y_n is $\sigma(X_1, \ldots, X_n)$-measurable for all n.
- (ii) $\mathrm{E}[|Y_n|] < \infty$ for all n.
- (iii) $\mathrm{E}[Y_{n+1} | X_1, \ldots, X_n] = Y_n$ a.s. for all n.

Remark:

- (i) just states that "Y_n is a function of $X_1, \ldots, X_n$ only".

Definitions and basic comments

A lot of examples...

Example 1

Let $X_1, X_2, \ldots$ be $\perp\!\!\!\perp$ integrable r.v.'s, with common mean 0.
Let $Y_n := \sum_{i=1}^{n} X_i$.
Then $\{Y_n\}$ is a martingale w.r.t. $\{X_n\}$.

Indeed,

- (i) is trivial.
- (ii) is trivial.
- (iii): with $\mathcal{A}_n := \sigma(X_1, \ldots, X_n)$, we have

$$\begin{aligned} \mathrm{E}[Y_{n+1}|\mathcal{A}_n] &= \mathrm{E}[Y_n + X_{n+1}|\mathcal{A}_n] = \mathrm{E}[Y_n|\mathcal{A}_n] + \mathrm{E}[X_{n+1}|\mathcal{A}_n] \\ &= Y_n + \mathrm{E}[X_{n+1}] = Y_n + 0 = Y_n \text{ a.s. for all } n, \end{aligned}$$

  where we used that Y_n is $\mathcal{A}_n$-measurable and that $X_{n+1} \perp\!\!\!\perp \mathcal{A}_n$.

Similarly, if $X_1, X_2, \ldots$ are $\perp\!\!\!\perp$ and integrable with means $\mu_1, \mu_2, \ldots$, respectively, $\{\sum_{i=1}^{n}(X_i - \mu_i)\}$ is a martingale w.r.t. $\{X_n\}$.

Example 2

Let $X_1, X_2, \dots$ be $\perp\!\!\!\perp$ integrable r.v.'s, with common mean 1.
Let $Y_n := \prod_{i=1}^{n} X_i$.
Then $\{Y_n\}$ is a martingale w.r.t. $\{X_n\}$.

Indeed,

- (i) is trivial.
- (ii) is trivial.
- (iii): with $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$, we have

  $$\mathrm{E}[Y_{n+1}|\mathcal{A}_n] = \mathrm{E}[Y_n X_{n+1}|\mathcal{A}_n] = Y_n \mathrm{E}[X_{n+1}|\mathcal{A}_n] = Y_n \mathrm{E}[X_{n+1}] = Y_n$$

  a.s. for all n, where we used that Y_n is $\mathcal{A}_n$-measurable (and hence behaves as a constant in $\mathrm{E}[\,.\,|\mathcal{A}_n]$).

Similarly, if $X_1, X_2, \dots$ are $\perp\!\!\!\perp$ and integrable with means $\mu_1, \mu_2, \dots$, respectively, $\{\prod_{i=1}^{n}(X_i/\mu_i)\}$ is a martingale w.r.t. $\{X_n\}$.

Example 2

The previous example is related to basic models for stock prices.

Assume $X_1, X_2, \ldots$ are positive, $\perp\!\!\!\perp$, and integrable r.v.'s.
Let $Y_n := c \prod_{i=1}^n X_i$, where c is the initial price.

The quantity $X_i - 1$ is the change in the value of the stock over a fixed time interval (one day, say) as a fraction of its current value. This multiplicative model (i) ensures nonnegativeness of the price and (ii) is compatible with the fact fluctuations in the value of a stock are roughly proportional to its price.

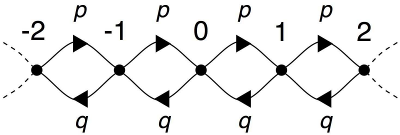
Various models are obtained for various distributions of the X_i's:

- Discrete Black-Scholes model: $X_i = e^{\eta_i}$, where $\eta_i \sim \mathcal{N}(\mu, \sigma^2)$ for all i.
- Binomial model: $X_i = e^{-r}(1+a)^{2\eta_i - 1}$, where $\eta_i \sim \mathrm{Bin}(1, p)$ for all i ($r =$ interest rate, by which one discounts future rewards).

Example 3: random walks

Let $X_1, X_2, \ldots$ be i.i.d., with $P[X_i = 1] = p$ and $P[X_i = -1] = q = 1 - p$
Let $Y_n := \sum_{i=1}^{n} X_i$.

$\rightsquigarrow$ The SP $\{Y_n\}$ is called a random walk.



Remarks:

- If $p = q$, the RW is said to be symmetric.
- Of course, from Example 1, we know that $\{(\sum_{i=1}^{n} X_i) - n(p-q)\}$ is a martingale w.r.t. $\{X_n\}$.

But other martingales exist for RWs...

Example 3: random walks

Consider the non-symmetric case ($p \neq q$) and let $S_n := \left(\frac{q}{p}\right)^{Y_n}$.

Then $\{S_n\}$ is a martingale w.r.t. $\{X_n\}$.

Indeed,

- (i) is trivial.
- (ii): $|S_n| \leq \max((q/p)^n, (q/p)^{-n})$. Hence, $\mathrm{E}[|S_n|] < \infty$.
- (iii): with $\mathcal{A}_n := \sigma(X_1, \ldots, X_n)$, we have

  $$\mathrm{E}[S_{n+1}|\mathcal{A}_n] = \mathrm{E}[(q/p)^{Y_n}(q/p)^{X_{n+1}}|\mathcal{A}_n] = (q/p)^{Y_n}\mathrm{E}[(q/p)^{X_{n+1}}|\mathcal{A}_n]$$
  $$= S_n\,\mathrm{E}[(q/p)^{X_{n+1}}] = S_n\left((q/p)^1 \times p + (q/p)^{-1} \times q\right) = S_n, \text{ a.s. for all}$$

  where we used that Y_n is $\mathcal{A}_n$-measurable and that $X_{n+1} \perp\!\!\!\perp \mathcal{A}_n$.

Example 3: random walks

Consider the symmetric case ($p = q$) and let $S_n := Y_n^2 - n$.

Then $\{S_n\}$ is a martingale w.r.t. $\{X_n\}$.

Indeed,

- (i) is trivial.
- (ii): $|S_n| \leq n^2 - n$. Hence, $\mathrm{E}[|S_n|] < \infty$.
- (iii): with $\mathcal{A}_n := \sigma(X_1, \ldots, X_n)$, we have

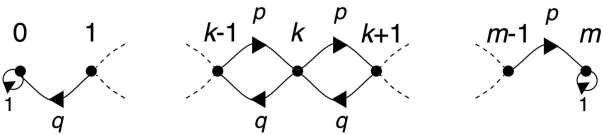
$$\begin{aligned}\mathrm{E}[S_{n+1}|\mathcal{A}_n] &= \mathrm{E}[(Y_n + X_{n+1})^2 - (n+1)|\mathcal{A}_n] \\ &= \mathrm{E}[(Y_n^2 + X_{n+1}^2 + 2Y_nX_{n+1}) - (n+1)|\mathcal{A}_n] \\ &= Y_n^2 + \mathrm{E}[X_{n+1}^2|\mathcal{A}_n] + 2Y_n\mathrm{E}[X_{n+1}|\mathcal{A}_n] - (n+1) \\ &= Y_n^2 + \mathrm{E}[X_{n+1}^2] + 2Y_n\mathrm{E}[X_{n+1}] - (n+1) \\ &= S_n \text{ a.s. for all } n,\end{aligned}$$

where we used that Y_n is $\mathcal{A}_n$-measurable and that $X_{n+1} \perp\!\!\!\perp \mathcal{A}_n$.

Example 4: De Moivre's martingales

Let $X_1, X_2, \ldots$ be i.i.d., with $P[X_i = 1] = p$ and $P[X_i = -1] = q = 1 - p$
Let $Y_0 := k \in \{1, 2, \ldots, m-1\}$ be the initial state.
Let $Y_{n+1} := (Y_n + X_{n+1})\, I[Y_n \notin \{0, m\}] + Y_n\, I[Y_n \in \{0, m\}]$.

$\rightsquigarrow$ The SP $\{Y_n\}$ is called a random walk with absorbing barriers.



Remarks:

- Before being caught either in 0 or m, this is just a RW.
- As soon as you get in 0 or m, you stay there forever.

Example 4: De Moivre's martingales

Let $X_1, X_2, \ldots$ be i.i.d., with $P[X_i = 1] = p$ and $P[X_i = -1] = q = 1 - p$
Let $Y_0 := k \in \{1, 2, \ldots, m-1\}$ be the initial state.
Let $Y_{n+1} := (Y_n + X_{n+1})\, I[Y_n \notin \{0, m\}] + Y_n\, I[Y_n \in \{0, m\}]$.

In this new setup and with this new definition of Y_n,

- In the non-symmetric case,
  $\{S_n := \left(\frac{q}{p}\right)^{Y_n}\}$ is still a martingale w.r.t. $\{X_n\}$.
- In the symmetric case,
  $\{S_n := Y_n^2 - n\}$ is still a martingale w.r.t. $\{X_n\}$.

(exercise).

Example 5: branching processes

Consider some population, in which each individual i of the Z_n individuals in the nth generation gives birth to $X_{n+1,i}$ children (the $X_{n,i}$'s are i.i.d., take values in $\mathbb{N}$, and have common mean $\mu < \infty$).

Assume that $Z_0 = 1$.

Then $\{Z_n/\mu^n\}$ is a martingale w.r.t. $\{\mathcal{A}_n := \sigma(\text{the } X_{m,i}\text{'s}, m \leq n)\}$.

Indeed,

- (i), (ii): exercise...
- (iii): $\mathrm{E}\left[\frac{Z_{n+1}}{\mu^{n+1}}|\mathcal{A}_n\right] = \frac{1}{\mu^{n+1}}\mathrm{E}\left[\sum_{i=1}^{Z_n} X_{n+1,i}|\mathcal{A}_n\right]$ $=\frac{1}{\mu^{n+1}}\sum_{i=1}^{Z_n}\mathrm{E}[X_{n+1,i}|\mathcal{A}_n] = \frac{1}{\mu^{n+1}}\sum_{i=1}^{Z_n}\mathrm{E}[X_{n+1,i}] = \frac{Z_n}{\mu^n}$ a.s. for all n.

In particular, $\mathrm{E}\left[\frac{Z_n}{\mu^n}\right] = \mathrm{E}\left[\frac{Z_0}{\mu^0}\right] = 1$. Hence, $\mathrm{E}[Z_n] = \mu^n$ for all n.

Example 6: Polya's urn

Consider an urn containing b blue balls and r red ones.
Pick randomly some ball in the urn and put it back in the urn with an extra ball of the same color. Repeat this procedure.

This is a so-called contamination process.

Let X_n be the number of red balls in the urn after n steps.
Let

$$R_n = \frac{X_n}{b+r+n}$$

be the proportion of red balls in the urn after n steps.

Then $\{R_n\}$ is a martingale w.r.t. $\{X_n\}$.

Example 6: Polya's urn

Indeed,

- (i) is trivial.
- (ii): $0 \leq |R_n| \leq 1$. Hence, $\mathrm{E}[|R_n|] < \infty$.
- (iii): with $\mathcal{A}_n := \sigma(X_1, \ldots, X_n)$, we have

$$
\begin{aligned}
\mathrm{E}[X_{n+1}|\mathcal{A}_n] &= (X_n + 1)\,\frac{X_n}{r+b+n} + (X_n + 0)\Big(1 - \frac{X_n}{r+b+n}\Big) \\
&= \frac{(X_n+1)X_n + X_n((r+b+n) - X_n)}{r+b+n} \\
&= \frac{(r+b+n+1)X_n}{r+b+n} \\
&= (r+b+n+1)R_n \text{ a.s. for all } n,
\end{aligned}
$$

so that $\mathrm{E}[R_{n+1}|\mathcal{A}_n] = R_n$ a.s. for all n.

Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
    - 3.1. Definitions and examples.
    - 3.2. Stopping times and the optional stopping theorem.
    - 3.3. Sub- and super- martingales, Limiting results.
4. Markov chains.
5. Markov processes, Poisson processes.
6. Brownian motions.

Stopping times

Let (Y_n) be a martingale w.r.t. $(\mathcal{A}_n)$
Let $T : (\Omega, \mathcal{A}, \mathbb{P}) \to \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ be a r.v.

Definition: T is a stopping time w.r.t. $(\mathcal{A}_n) \Leftrightarrow$
(i) T is a.s. finite (i.e., $\mathbb{P}[T < \infty] = 1$).
(ii) $[T = n] \in \mathcal{A}_n$ for all n.

Remarks:

- (ii) is the crucial assumption:
  it says that one knows, at time n, on the basis of the "information" $\mathcal{A}_n$, whether $T = n$ or not, that is, whether one should stop at n or not.
- (i) just makes (almost) sure that one will stop at some point.

Stopping times (examples)

(Kind of) examples...

Let (Y_n) be a martingale w.r.t. $(\mathcal{A}_n)$. Let $B \in \mathcal{B}$.

(A) Let $T := \inf\{n \in \mathbb{N} | Y_n \in B\}$ be the time of 1st entry of (Y_n) into B. Then,

$$[T = n] = [Y_0 \notin B, Y_1 \notin B, \ldots, Y_{n-1} \notin B, Y_n \in B] \in \mathcal{A}_n.$$

Hence, provided that T is a.s. finite, T is a ST.

(B) Let $T := \sup\{n \in \mathbb{N} | Y_n \in B\}$ be the time of last escape of (Y_n) out of B. Then,

$$[T = n] = [Y_n \in B, Y_{n+1} \notin B, Y_{n+2} \notin B, \ldots] \notin \mathcal{A}_n.$$

Hence, T is not a ST.

Stopping times (examples)

(C) Let $T := k$ a.s. (for some fixed integer k). Then, of course, (i) $T < \infty$ a.s. and (ii)

$$[T = n] = \begin{cases} \emptyset & \text{if} \quad n \neq k \\ \Omega & \text{if} \quad n = k, \end{cases}$$

which is in $\mathcal{A}_n$ for all n. Hence, T is a ST.

Stopping times (properties)

Properties:

- $[T = n] \in \mathcal{A}_n \; \forall n \overset{(1)}{\Leftrightarrow} [T \leq n] \in \mathcal{A}_n \; \forall n \overset{(2)}{\Leftrightarrow} [T > n] \in \mathcal{A}_n \; \forall n$.

  Indeed,

  $\overset{(1)}{\Rightarrow}$ follows from $[T \leq n] = \bigcup_{k=1}^{n} [T = k]$.

  $\overset{(1)}{\Leftarrow}$ follows from $[T = n] = [T \leq n] \backslash [T \leq n-1]$.

  $\overset{(2)}{\Leftrightarrow}$ follows from $[T \leq n] = \Omega \backslash [T > n]$.

- T_1, T_2 are ST $\Rightarrow T_1 + T_2$, $\max(T_1, T_2)$, and $\min(T_1, T_2)$ are ST (exercise).

- Let (Y_n) be a martingale w.r.t. $(\mathcal{A}_n)$.
  Let T be a ST w.r.t. $(\mathcal{A}_n)$.
  Then $Y_T := \sum_{n=0}^{\infty} Y_n \mathbb{I}_{[T=n]}$ is a r.v.

  Indeed, $[Y_T \in B] = \bigcup_{n=0}^{\infty} \{[T = n] \cap [Y_n \in B]\} \in \mathcal{A}$.

Stopped martingale

A key lemma:

Lemma: Let (Y_n) be a martingale w.r.t. $(\mathcal{A}_n)$. Let T be a ST w.r.t. $(\mathcal{A}_n)$.
Then $\{Z_n := Y_{\min(n,T)}\}$ is a martingale w.r.t. $(\mathcal{A}_n)$.

Proof: note that

$$Z_n = Y_{\min(n,T)} = \sum_{k=0}^{n-1} Y_k \mathbb{I}_{[T=k]} + Y_n \mathbb{I}_{[T \geq n]}.$$

So

- (i): Z_n is $\mathcal{A}_n$-measurable for all n.
- (ii): $|Z_n| \leq \sum_{k=0}^{n-1} |Y_k| \mathbb{I}_{[T=k]} + |Y_n| \mathbb{I}_{[T \geq n]} \leq \sum_{k=0}^{n} |Y_k|$. Hence, $\mathbb{E}[|Z_n|] < \infty$.

Stopped martingale

- (iii): we have

$$\begin{aligned}\mathbb{E}[Z_{n+1}|\mathcal{A}_n] - Z_n &= \mathbb{E}[Z_{n+1} - Z_n|\mathcal{A}_n] \\ &= \mathbb{E}[(Y_{n+1} - Y_n)\mathbb{I}_{[T\geq n+1]}|\mathcal{A}_n] \\ &= \mathbb{E}[(Y_{n+1} - Y_n)\mathbb{I}_{[T>n]}|\mathcal{A}_n] \\ &= \mathbb{I}_{[T>n]}\,\mathbb{E}[Y_{n+1} - Y_n|\mathcal{A}_n] \\ &= \mathbb{I}_{[T>n]}\left(\mathbb{E}[Y_{n+1}|\mathcal{A}_n] - Y_n\right) \\ &= 0 \ \text{ a.s. for all } n,\end{aligned}$$

where we used that $\mathbb{I}_{[T>n]}$ is $\mathcal{A}_n$-measurable. $\square$

Stopped martingale

Corollary: Let (Y_n) be a martingale w.r.t. $(\mathcal{A}_n)$. Let T be a ST w.r.t. $(\mathcal{A}_n)$. Then $\mathbb{E}[Y_{\min(n,T)}] = \mathbb{E}[Y_0]$ for all n.

Proof: the lemma and the mean-stationarity of martingales yield $\mathbb{E}[Y_{\min(n,T)}] = \mathbb{E}[Z_n] = \mathbb{E}[Z_0] = \mathbb{E}[Y_{\min(0,T)}] = \mathbb{E}[Y_0]$ for all n. $\square$

In particular, if the ST is such that $T \leq k$ a.s. for some k, we have that, for $n \geq k$,

$$Y_{\min(n,T)} = Y_T \text{ a.s.},$$

so that

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0].$$

Stopped martingale

$\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$ does not always hold.

Example: the doubling strategy, for which the winnings are

$$Y_n = \sum_{i=1}^{n} C_i X_i,$$

where the X_i's are i.i.d. $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}$ and $C_i = 2^{i-1}b$.

The SP (Y_n) is a martingale w.r.t. (X_n) (exercise).
Let $T = \inf\{n \in \mathbb{N} | X_n = 1\}$ (exercise: T is a ST).

As we have seen, $Y_T = b$ a.s., so that $\mathbb{E}[Y_T] = b \neq 0 = \mathbb{E}[Y_0]$.

However, as shown by the following result, $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$ holds under much broader conditions than "$T \leq k$ a.s."

Optional stopping theorem

Theorem: Let (Y_n) be a martingale w.r.t. $(\mathcal{A}_n)$. Let T be a ST w.r.t. $(\mathcal{A}_n)$. Then if (i) $\mathbb{E}[|Y_T|] < \infty$ and (ii) $\lim_{n\to\infty} \mathbb{E}[Y_n \mathbb{I}_{[T>n]}] = 0$, we have $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$.

Proof: since $Y_{\min(n,T)} = Y_n \mathbb{I}_{[T>n]} + Y_T \mathbb{I}_{[T\leq n]}$, we have

$$Y_T = Y_T \mathbb{I}_{[T\leq n]} + Y_T \mathbb{I}_{[T>n]} = (Y_{\min(n,T)} - Y_n \mathbb{I}_{[T>n]}) + Y_T \mathbb{I}_{[T>n]}.$$

Taking expectations, we obtain

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0] - \mathbb{E}[Y_n \mathbb{I}_{[T>n]}] + \mathbb{E}[Y_T \mathbb{I}_{[T>n]}].$$

By taking the limit as $n \to \infty$ and using (ii),

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0] + \lim_{n\to\infty} \mathbb{E}[Y_T \mathbb{I}_{[T>n]}].$$

The result follows from $\lim_{n\to\infty} \mathbb{P}[T > n] = \mathbb{P}[T = \infty] = 0$. $\square$

Optional stopping theorem

Theorem: Let (Y_n) be a martingale w.r.t. $(\mathcal{A}_n)$. Let T be a ST w.r.t. $(\mathcal{A}_n)$. Then if (i) $\mathbb{E}[|Y_T|] < \infty$ and (ii) $\lim_{n\to\infty} \mathbb{E}[Y_n \mathbb{I}_{[T>n]}] = 0$, we have $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$.

Particular sufficient conditions for (i), (ii):

- (a) $T \leq k$ a.s. Indeed,

  (i) $\mathbb{E}[|Y_T|] = \mathbb{E}[|\sum_{n=0}^{k} Y_n \mathbb{I}_{[T=n]}|] \leq \sum_{n=0}^{k} \mathbb{E}[|Y_n|] < \infty$.
  (ii) $Y_n \mathbb{I}_{[T>n]} = 0$ a.s. for $n > k$. Hence, $\mathbb{E}[Y_n \mathbb{I}_{[T>n]}] = 0$ for $n > k$, so that (ii) holds.

- (b) (Y_n) is uniformly integrable.

Optional stopping theorem (examples)

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$.
Let $Y_0 := k \in \{1, 2, \ldots, m-1\}$ be the initial state.
Let $Y_{n+1} := (Y_n + X_{n+1})\, \mathbb{I}_{[Y_n \notin \{0,m\}]} + Y_n\, \mathbb{I}_{[Y_n \in \{0,m\}]}$.

$\rightsquigarrow$ The SP (Y_n) is called a random walk with absorbing barriers.

In the symmetric case, (Y_n) is a martingale w.r.t. (X_n) (exercise).
Let $T := \inf\{n \in \mathbb{N} | Y_n \in \{0, m\}\}$ (exercise: T is a stopping time, and the assumptions of the optional stopping thm are satisfied).

$\rightsquigarrow$ $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$.

Optional stopping theorem (examples)

Let $p_k := \mathbb{P}[Y_T = 0]$.

Then

$$\mathbb{E}[Y_T] = 0 \times p_k + m \times (1 - p_k)$$

and

$$\mathbb{E}[Y_0] = \mathbb{E}[k] = k,$$

so that $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$ yields

$$m(1 - p_k) = k,$$

that is, solving for p_k,

$$p_k = \frac{m - k}{m}.$$

Optional stopping theorem (examples)

Is there a way to get $\mathbb{E}[T]$ (still in the symmetric case)?

We know that $(S_n := Y_n^2 - n)$ is also a martingale w.r.t. (X_n) (exercise: with this martingale and the same ST, the assumptions of the optional stopping theorem are still satisfied).

$\rightsquigarrow$ $\mathbb{E}[S_T] = \mathbb{E}[S_0]$,where

$$\mathbb{E}[S_T] = \mathbb{E}[Y_T^2] - \mathbb{E}[T] = \Big(0^2 \times p_k + m^2 \times (1 - p_k)\Big) - \mathbb{E}[T]$$

and

$$\mathbb{E}[S_0] = \mathbb{E}[Y_0^2 - 0] = \mathbb{E}[k^2] = k^2.$$

Hence,

$$\mathbb{E}[T] = m^2(1 - p_k) - k^2 = m^2 \times \frac{k}{m} - k^2 = k(m - k).$$

Optional stopping theorem (examples)

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$.
Let $Y_0 := k \in \{1, 2, \ldots, m-1\}$ be the initial state.
Let $Y_{n+1} := (Y_n + X_{n+1})\, \mathbb{I}_{[Y_n \notin \{0,m\}]} + Y_n\, \mathbb{I}_{[Y_n \in \{0,m\}]}$.

$\rightsquigarrow$ The SP (Y_n) is called a random walk with absorbing barriers.

In the non-symmetric case, $\left(S_n := \left(\frac{q}{p}\right)^{Y_n}\right)$ is a martingale w.r.t. (X_n). Let $T := \inf\{n \in \mathbb{N} | S_n \in \{0, m\}\}$ (exercise: T is a stopping time, and the assumptions of the optional stopping thm are satisfied). $\rightsquigarrow$ $\mathbb{E}[S_T] = \mathbb{E}[S_0]$.

Optional stopping theorem (examples)

Let again $p_k := \mathbb{P}[Y_T = 0]$. Then, since

$$\mathbb{E}[S_T] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{Y_T}\right] = \left(\frac{q}{p}\right)^0 \times p_k + \left(\frac{q}{p}\right)^m (1 - p_k),$$

and

$$\mathbb{E}[S_0] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{Y_0}\right] = \mathbb{E}\left[\left(\frac{q}{p}\right)^k\right] = \left(\frac{q}{p}\right)^k,$$

we deduce that

$$\left(\frac{q}{p}\right)^0 \times p_k + \left(\frac{q}{p}\right)^m (1 - p_k) = \left(\frac{q}{p}\right)^k,$$

that is, solving for p_k,

$$p_k = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^m}.$$

Optional stopping theorem (examples)

Is there a way to get $\mathbb{E}[T]$ here as well?

In the non-symmetric case, $(R_n := Y_n - \min(n, T)(p-q))$ is a martingale w.r.t. (X_n) (exercise: check this, and check that the optional stopping thm applies with (R_n) and T).

$\rightsquigarrow$ $\mathbb{E}[R_T] = \mathbb{E}[R_0]$, where

$$\mathbb{E}[R_T] = \mathbb{E}[Y_T - T(p-q)] = \mathbb{E}[Y_T] - (p-q)\mathbb{E}[T] = \Big(0 \times p_k + m \times (1-p_k)\Big) -$$

and

$$\mathbb{E}[R_0] = \mathbb{E}[Y_0 - \min(0, T)(p-q)] = \mathbb{E}[Y_0] = \mathbb{E}[k] = k.$$

Hence,

$$\mathbb{E}[T] = \frac{m(1-p_k) - k}{p-q} = \frac{m\left(1 - \left(\frac{q}{p}\right)^k\right) - k\left(1 - \left(\frac{q}{p}\right)^m\right)}{(p-q)\left(1 - \left(\frac{q}{p}\right)^m\right)}.$$

Outline of the course

1. A short introduction.
2. Basic probability review.
3. $\boxed{\text{3. Martingales.}}$
    3.1. Definitions and examples.
    3.2. Stopping times and the optional stopping theorem.
    3.3. $\boxed{\text{3.3. Sub- and super- martingales, Limiting results.}}$
4. Markov chains.
5. Markov processes, Poisson processes.
6. Brownian motions.

Sub- and super-martingales

Not every game is fair...
There are also favourable and defavourable games.

Therefore, we introduce the following concepts:

The SP $(Y_n)_{n\in\mathbb{N}}$ is a submartingale w.r.t. the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}} \Leftrightarrow$

- (i) $(Y_n)_{n\in\mathbb{N}}$ is adapted to the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$.
- (ii)' $\mathbb{E}[Y_n^+] < \infty$ for all n.
- (iii)' $\mathbb{E}[Y_{n+1}|\mathcal{A}_n] \geq Y_n$ a.s. for all n.

The SP $(Y_n)_{n\in\mathbb{N}}$ is a supermartingale w.r.t. the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}} \Leftrightarrow$

- (i) $(Y_n)_{n\in\mathbb{N}}$ is adapted to the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$.
- (ii)" $\mathbb{E}[Y_n^-] < \infty$ for all n.
- (iii)" $\mathbb{E}[Y_{n+1}|\mathcal{A}_n] \leq Y_n$ a.s. for all n.

Sub- and super-martingales

Remarks:

- (iii)' shows that a submartingale can be thought of as the fortune of a gambler betting on a favourable game.
- (iii)' $\Rightarrow \mathbb{E}[Y_n] \geq \mathbb{E}[Y_0]$ for all n.

- (iii)" shows that a supermartingale can be thought of as the fortune of a gambler betting on a defavourable game.
- (iii)" $\Rightarrow \mathbb{E}[Y_n] \leq \mathbb{E}[Y_0]$ for all n.

- (Y_n) is a submartingale w.r.t. $(\mathcal{A}_n)$
  $\Leftrightarrow (-Y_n)$ is a supermartingale w.r.t. $(\mathcal{A}_n)$.
- (Y_n) is a martingale w.r.t. $(\mathcal{A}_n)$
  $\Leftrightarrow (Y_n)$ is both a sub- and a supermartingale w.r.t. $(\mathcal{A}_n)$.

Sub- and super-martingales

Consider the following strategy for the fair version of roulette (without the "0" slot):

Bet b euros on an even result. If you win, stop.
If you lose, bet $2b$ euros on an even result. If you win, stop.
If you lose, bet $4b$ euros on an even result. If you win, stop...
And so on...

How good is this strategy?

(a) If you first win in the nth game, your total winning is

$$-\sum_{i=0}^{n-2} 2^i b + 2^{n-1} b = b$$

$\Rightarrow$ Whatever the value of n is, you win b euros with this strategy.

Sub- and super-martingales

(b) You will a.s. win. Indeed, let T be the time index of first success. Then

$$\mathbb{P}[T < \infty] = \sum_{n=1}^{\infty} \mathbb{P}[n-1 \text{ first results are "odd", then "even"}]$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} = 1.$$

But

(c) The expected amount you lose just before you win is

$$0 \times \frac{1}{2} + b \times \left(\frac{1}{2}\right)^2 + (b+2b) \times \left(\frac{1}{2}\right)^3 + \ldots + \left(\sum_{i=0}^{n-2} 2^i b\right) \left(\frac{1}{2}\right)^n + \ldots = \infty$$

$\Rightarrow$ Your expected loss is infinite!

(d) You need an unbounded wallet...

Sub- and super-martingales

Let us try to formalize strategies...

Consider the SP $(X_n)_{n\in\mathbb{N}}$, where X_n is your winning per unit stake in game n. Denote by $(\mathcal{A}_n)_{n\in\mathbb{N}}$ the corresponding filtration ($\mathcal{A}_n = \sigma(X_1, \ldots, X_n)$).

Definition: A gambling strategy (w.r.t. (X_n)) is a SP $(C_n)_{n\in\mathbb{N}}$ such that C_n is $\mathcal{A}_{n-1}$-measurable for all n.

Remarks:

- $C_n = C_n(X_1, \ldots, X_{n-1})$ is what you will bet in game n.
- $\mathcal{A}_0 = \{\emptyset, \Omega)$.

Sub- and super-martingales

Using some strategy (C_n), your total winning after n games is

$$Y_n^{(C)} = \sum_{i=1}^{n} C_i X_i.$$

A natural question:
Is there any way to choose (C_n) so that $(Y_n^{(C)})$ is "nice"?.

Consider the "blind" strategy $C_n = 1$ (for all n), that consists in betting 1 euro in each game, and denote by $(Y_n = \sum_{i=1}^{n} X_i)$ the corresponding process of winnings.

Then, here is the answer:

Theorem: Let (C_n) be a gambling strategy with nonnegative and bounded r.v.'s. Then if (Y_n) is a martingale, so is $(Y_n^{(C)})$. If (Y_n) is a submart., so is $(Y_n^{(C)})$. And if (Y_n) is a supermart., so is $(Y_n^{(C)})$.

Sub- and super-martingales

Proof:

- (i) is trivial.
- (ii): $|Y_n^{(C)}| \leq \sum_{i=1}^n a_i |X_i|$. Hence, $\mathbb{E}[|Y_n^{(C)}|] < \infty$.
- (iii),(iii)',(iii)": with $\mathcal{A}_n := \sigma(X_1, \ldots, X_n)$, we have

$$\begin{aligned}
\mathbb{E}[Y_{n+1}^{(C)}|\mathcal{A}_n] &= \mathbb{E}[Y_n^{(C)} + C_{n+1}X_{n+1}|\mathcal{A}_n] \\
&= Y_n^{(C)} + C_{n+1}\,\mathbb{E}[X_{n+1}|\mathcal{A}_n] \\
&= Y_n^{(C)} + C_{n+1}\,\mathbb{E}[Y_{n+1} - Y_n|\mathcal{A}_n] \\
&= Y_n^{(C)} + C_{n+1}\left(\mathbb{E}[Y_{n+1}|\mathcal{A}_n] - Y_n\right),
\end{aligned}$$

where we used that C_{n+1} is $\mathcal{A}_n$-measurable. Since $C_{n+1} \geq 0$, the result follows. $\square$

Remark: The second part was checked for martingales only.
Exercise: check (ii)' and (ii)"...

Convergence of martingales

Theorem: let (Y_n) be a submartingale w.r.t. $(\mathcal{A}_n)$. Assume that, for some M, $\mathbb{E}[Y_n^+] \leq M$ for all n. Then
(i) $\exists Y_\infty$ such that $Y_n \overset{a.s.}{\to} Y_\infty$ as $n \to \infty$.
(ii) If $\mathbb{E}[|Y_0|] < \infty$, $\mathbb{E}[|Y_\infty|] < \infty$.

The following results directly follow:

Corollary 1: let (Y_n) be a submartingale or a supermartingale w.r.t. $(\mathcal{A}_n)$. Assume that, for some M, $\mathbb{E}[|Y_n|] \leq M$ for all n. Then $\exists Y_\infty$ (satisfying $\mathbb{E}[|Y_\infty|] < \infty$) such that $Y_n \overset{a.s.}{\to} Y_\infty$ as $n \to \infty$.

Corollary 2: let (Y_n) be a negative submartingale or a positive supermartingale w.r.t. $(\mathcal{A}_n)$. Then $\exists Y_\infty$ such that $Y_n \overset{a.s.}{\to} Y_\infty$ as $n \to \infty$.

Example: products of r.v.'s

Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s, with common distribution

distribution of X_i		
values	0	2
probabilities	$\frac{1}{2}$	$\frac{1}{2}$

The X_i's are integrable r.v.'s with common mean 1, so that $(Y_n = \prod_{i=1}^{n} X_i)$ is a (positive) martingale w.r.t. (X_n) (example 2 in the previous lecture).

Consequently, $\exists\, Y_\infty$ such that $Y_n \xrightarrow{a.s.} Y_\infty$ as $n \to \infty$.
We showed, in Lecture #4, that $Y_n \xrightarrow{P} 0$ as $n \to \infty$ so that $Y_\infty = 0$ a.s.

But we also showed there that convergence in L^1 does not hold. To ensure L^1-convergence, one has to require uniform integrability of (Y_n).

Example: Polya's urn

Consider an urn containing b blue balls and r red ones. Pick randomly some ball in the urn and put it back in the urn with an extra ball of the same color. Repeat this procedure.

Let X_n be the number of red balls in the urn after n steps. Let $R_n = \frac{X_n}{b+r+n}$ be the proportion of red balls after n steps.

We know that (R_n) is a martingale w.r.t. (X_n).

Now, $|R_n| \leq 1$ ($\Rightarrow \mathbb{E}[|R_n|] \leq 1$), so that $\exists R_\infty$ (satisfying $\mathbb{E}[|R_\infty|] < \infty$) such that $R_n \xrightarrow{a.s.} R_\infty$ as $n \to \infty$.

Clearly, uniform integrability holds. Hence, $R_n \xrightarrow{L^1} R_\infty$ as $n \to \infty$.

Remark: it can be shown that R_∞ has a beta distribution:

$$\mathbb{P}[R_\infty \leq u] = \binom{b+r}{r} \int_0^u x^{r-1}(1-x)^{b-1}\,dx, \quad u \in (0,1).$$

Example: branching processes

Consider some population, in which each individual i of the Z_n individuals in the nth generation gives birth to $X_{n,i}$ children (the $X_{n,i}$'s are i.i.d., take values in $\mathbb{N}$, and have common mean $\mu < \infty$).

Assume that $Z_0 = 1$.

Then $(Y_n := Z_n/\mu^n)$ is a martingale w.r.t.
$(\mathcal{A}_n := \sigma(X_{n,i}, n \text{ fixed}))$.

What can be said about the long-run state?

$\mathbb{E}[|Y_n|] = \mathbb{E}[Y_0] = 1$ for all $n \Rightarrow \exists Y_\infty$ (satisfying $\mathbb{E}[|Y_\infty|] < \infty$) such that $Y_n \stackrel{a.s.}{\to} Y_\infty$ as $n \to \infty$.

$\rightsquigarrow Z_n \approx Y_\infty \mu^n$ for large n.

Example: branching processes

Case 1: $\mu \leq 1$.

If $p_0 := \mathbb{P}[X_{n,i} = 0] > 0$, $Z_n \overset{a.s.}{\to} Z_\infty := 0$ as $n \to \infty$.

Proof:

Let $k \in \mathbb{N}_0$. Then

$$\mathbb{P}[Z_\infty = k] = \mathbb{P}[Z_n = k \ \forall n \geq n_0] = \mathbb{P}[Z_{n_0} = k] \left(\mathbb{P}[Z_{n_0+1} = k | Z_{n_0} = k]\right)^\infty$$

where

$$\mathbb{P}[Z_{n_0+1} = k | Z_{n_0} = k] \leq \mathbb{P}[Z_{n_0+1} \neq 0 | Z_{n_0} = k] = 1 - (p_0)^k < 1.$$

Therefore, $\mathbb{P}[Z_\infty = k] = 0$ for all $k \in \mathbb{N}_0$, so that $Z_\infty = 0$ a.s. $\square$

Example: branching processes

Case 2: $\mu > 1$.

Then $Z_n \overset{a.s.}{\to} Z_\infty$ as $n \to \infty$, where the distribution of Z_∞ is given by

distribution of Z_∞		
values	0	∞
probabilities	π	$1-\pi$

where $\pi \in (0,1)$ is the unique solution of $g(s) = s$, where, defining $p_k := \mathbb{P}[X_{n,i} = k]$, we let $g(s) := \sum_{k=0}^{\infty} p_k s^k$.

Proof:

Clearly, since $Z_n \approx Y_\infty \mu^n$ for large n, $Z_n \overset{a.s.}{\to} Z_\infty$ as $n \to \infty$, where Z_∞ only assumes the values 0 and ∞, with proba a and $1-a$, respectively, say.

We may assume that $p_0 > 0$ (clearly, if $p_0 = 0$, $Z_\infty = \infty$ a.s.)

Example: branching processes

There is indeed a unique $\pi \in (0,1)$ such that $g(\pi) = \pi$, since g is monotone increasing, $g(0) = p_0 > 0$, and $g'(1) = \mu > 1$.

$\rightsquigarrow$ (π^{Z_n}) is a martingale w.r.t. $(\mathcal{A}_n)$, since

$$\mathbb{E}[\pi^{Z_{n+1}}|\mathcal{A}_n] = \mathbb{E}[\pi^{\sum_{i=1}^{Z_n} X_{n+1,i}}|\mathcal{A}_n] = \left(\mathbb{E}[\pi^{X_{n+1,1}}|\mathcal{A}_n]\right)^{Z_n}$$

$$= \left(\mathbb{E}[\pi^{X_{n+1,1}}]\right)^{Z_n} = \left(\sum_{k=0}^{\infty} \pi^k p_k\right)^{Z_n} = \left(g(\pi)\right)^{Z_n} = \pi^{Z_n}$$

a.s. for all n.

(π^{Z_n}) is also uniformly integrable, so that $\pi^{Z_n} \xrightarrow{L^1} \pi^{Z_\infty}$ as $n \to \infty$. Therefore, $\mathbb{E}[\pi^{Z_\infty}] = \pi^0 \times a + \pi^\infty \times (1-a) = a$ is equal to $\mathbb{E}[\pi^{Z_0}] = \mathbb{E}[\pi^1] = \pi$.

Hence, $\mathbb{P}[Z_\infty = 0] = \pi$ and $\mathbb{P}[Z_\infty = \infty] = 1 - \pi$. $\square$

Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
4. $\boxed{\text{4. Markov chains.}}$
    - 4.1. Definitions and examples.
    - 4.2. Strong Markov property, number of visits.
    - 4.3. Classification of states.
    - 4.4. Computation of R and F.
    - 4.5. Asymptotic behavior.
5. Markov processes, Poisson processes.
6. Brownian motions.

Markov chains

The importance of these processes comes from two facts:

- there is a large number of physical, biological, economic, and social phenomena that can be described in this way, and
- there is a well-developed theory that allows for doing the computations and obtaining explicit results...

Definitions and examples

Let S be a finite or countable set (number its elements using $i = 1, 2, .$
Let $(X_n)_{n \in \mathbb{N}}$ be a SP with $X_n : (\Omega, \mathcal{A}, P) \to S$ for all n.

Definition: (X_n) is a Markov chain (MC) on $S \Leftrightarrow$

$$\mathbb{P}[X_{n+1} = j | X_0, X_1, \ldots, X_n] = \mathbb{P}[X_{n+1} = j | X_n] \quad \forall n \, \forall j.$$

Remarks:

- The equation above is the so-called Markov property. It states that the future does only depend on the present state of the process, and not on its past.
- S is the state space.
- The elements of S are the states.

Definition: The MC (X_n) is homogeneous ($\rightsquigarrow$ HMC) $\Leftrightarrow$

$$\mathbb{P}[X_{n+1} = j | X_n = i] = \mathbb{P}[X_1 = j | X_0 = i] \quad \forall n \, \forall i, j.$$

For a HMC, one can define the transition probabilities

$$p_{ij} = \mathbb{P}[X_{n+1} = j | X_n = i] \quad \forall i, j,$$

which are usually collected in the transition matrix $P = (p_{ij})$.

The transition matrix P is a "stochastic matrix", which means that

- $p_{ij} \in [0, 1]$ for all i, j.
- $\sum_j p_{ij} = 1$ for all i.

In vector notation, $P1 = 1$, where 1 stands for the vector of ones with the appropriate dimension.

Example 1: random walk

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$
Let $Y_i := \sum_{i=1}^{n} X_i$ be the corresponding random walk.

$\rightsquigarrow$ (Y_n) is a HMC on $S = \mathbb{Z}$ with transition matrix

$$P = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & q & 0 & p & & & \\ & & q & 0 & p & & \\ & & & q & 0 & p & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Example 2: rw with absorbing barriers

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$
Let $Y_0 := k \in \{1, 2, \ldots, m-1\}$ be the initial state.
Let $Y_{n+1} := (Y_n + X_{n+1})\, \mathbb{I}_{[Y_n \notin \{0,m\}]} + Y_n\, \mathbb{I}_{[Y_n \in \{0,m\}]}$.

$\rightsquigarrow$ (Y_n) is a HMC on $S = \{0, 1, \ldots, m\}$ with transition matrix

$$P = \begin{pmatrix} 1 & 0 & \ldots & & & & \\ q & 0 & p & & & & \\ & q & 0 & p & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & q & 0 & p \\ & & & & \ldots & 0 & 1 \end{pmatrix}.$$

Example 3: success runs

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = 0] = q = 1 - p$.
Let $Y_0 := 0$ be the initial state.
Let $Y_{n+1} := (Y_n + 1)\, \mathbb{I}_{[X_{n+1}=1]} + 0 \times \mathbb{I}_{[X_{n+1}=0]}$.

$\rightsquigarrow$ (Y_n) is a HMC on $S = \mathbb{N}$ with transition matrix

$$P = \begin{pmatrix} q & p & 0 & \ldots & & \\ q & 0 & p & 0 & \ldots & \\ q & 0 & 0 & p & 0 & \ldots \\ \vdots & & & \ddots & \ddots & \end{pmatrix}.$$

Example 4: discrete queue models

Let Y_n be the number of clients in a queue at time n ($Y_0 = 0$). Let X_n be the number of clients entering the shop between time $n-1$ and n (X_n i.i.d., with $\mathbb{P}[X_n = i] = p_i$; $\sum_{i=0}^{\infty} p_i = 1$). Assume a service needs exactly one unit of time to be completed.

Then

$$Y_{n+1} = (Y_n + X_n - 1)\,\mathbb{I}_{[Y_n>0]} + X_n\,\mathbb{I}_{[Y_n=0]},$$

and (Y_n) is a HMC on $S = \mathbb{N}$ with transition matrix

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & \dots & & \\ p_0 & p_1 & p_2 & \dots & & \\ 0 & p_0 & p_1 & p_2 & \dots & \\ 0 & 0 & p_0 & p_1 & p_2 & \dots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Example 5: stock management

Let X_n be the number of units on hand at the end of day n ($X_0 = M$).
Let D_n be the demand on day n (D_n i.i.d., $\mathbb{P}[D_n = i] = p_i$; $\sum_{i=0}^{\infty} p_i = 1$).
Assume that if $X_n \leq m$, it is (instantaneously) set to M again.

Then, letting $x^+ = \max(x, 0)$, we have

$$X_{n+1} = (X_n - D_{n+1})^+ \, \mathbb{I}_{[X_n > m]} + (M - D_{n+1})^+ \, \mathbb{I}_{[X_n \leq m]},$$

and (X_n) is a HMC on $S = \{0, 1, \dots, M\}$ (exercise: derive P).

Questions:

- if we make 12\$ profit on each unit sold but it costs 2\$ a day to store items, what is the long-run profit per day of this inventory policy?
- How to choose (m, M) to maximize profit?

Example 6: income classes

Assume that from one generation to the next, families change their income group "Low", "Middle", or "High" (state 1,2, and 3, respectively) according to a HMC with transition matrix

$$P = \begin{pmatrix} .6 & .3 & .1 \\ .2 & .7 & .1 \\ .1 & .3 & .6 \end{pmatrix}.$$

Questions:

- Do the fractions of the population in the three income classes stabilize as time goes on?
- If this happens, how can we compute the limiting proportions from P?

Higher-order transition probabilities

We let $P = (p_{ij})$, where $p_{ij} = \mathbb{P}[X_1 = j | X_0 = i]$.
Now, define $P^{(n)} = (p_{ij}^{(n)})$, where $p_{ij}^{(n)} = \mathbb{P}[X_n = j | X_0 = i]$.

What is the link between P and $P^{(n)}$?

$\rightsquigarrow$ **Theorem**: $P^{(n)} = P^n$.

Proof: the result holds for $n = 1$. Now, assume it holds for n.
Then $(P^{(n+1)})_{ij} = \mathbb{P}[X_{n+1} = j | X_0 = i] = \sum_k \mathbb{P}[X_{n+1} = j, X_n = k | X_0 = i] = \sum_k \mathbb{P}[X_{n+1} = j | X_n = k, X_0 = i] \mathbb{P}[X_n = k | X_0 = i]$
$= \sum_k \mathbb{P}[X_{n+1} = j | X_n = k] \mathbb{P}[X_n = k | X_0 = i] =$
$\sum_k (P^{(n)})_{ik} (P^{(1)})_{kj} = (P^{(n)} P)_{ij} = (P^n P)_{ij} = (P^{n+1})_{ij}$, so that the result holds for $n + 1$ as well. $\square$

Higher-order transition probabilities

Of course, this implies that $P^{(n+m)} = P^{n+m} = P^n P^m = P^{(n)} P^{(m)}$, that is,

$$\mathbb{P}[X_{n+m} = j | X_0 = i] = \sum_k \mathbb{P}[X_n = k | X_0 = i] \mathbb{P}[X_m = j | X_0 = k],$$

which are the so-called Chapman-Kolmogorov equations.

Higher-order transition probabilities

Clearly, the distribution of X_n is of primary interest.

Let $a^{(n)}$ be the (line) vector with jth component $(a^{(n)})_j = \mathbb{P}[X_n = j]$.
$\rightsquigarrow$ **Theorem**: $a^{(n)} = a^{(0)} P^n$.

Proof: using the total probability formula, we obtain $(a^{(n)})_j = \mathbb{P}[X_n = j] = \sum_k \mathbb{P}[X_n = j | X_0 = k] \mathbb{P}[X_0 = k] = \sum_k (a^{(0)})_k (P^{(n)})_{kj} = (a^{(0)} P^{(n)})_j = (a^{(0)} P^n)_j$, which establishes the result. $\square$

This shows that one can very easily compute the distribution of X_n in terms of

- the distribution of X_0, and
- the transition matrix P.

Higher-order transition probabilities

Proposition: let (X_n) be a HMC on S. Then

$$\mathbb{P}[X_1 = i_1, X_2 = i_2, \dots, X_n = i_n | X_0 = i_0],$$
$$\mathbb{P}[X_{m+1} = i_1, X_{m+2} = i_2, \dots, X_{m+n} = i_n | X_m = i_0],$$

and

$$\mathbb{P}[X_{m+1} = i_1, X_{m+2} = i_2, \dots, X_{m+n} = i_n | X_0 = j_0, X_1 = j_1, \dots, X_m = i_0]$$

all are equal to $p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n}$.

Proof: exercise...