Stochastic Processes (Lecture #5)

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Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
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1. A short introduction.
2. Basic probability review.
3. **Martingales.**
   3.1. Definitions and examples.
   3.2. Sub- and super- martingales, gambling strategies.
   3.3. Stopping times and the optional stopping theorem.
   3.4. Limiting results.
Definitions and basic comments

Let \((\Omega, \mathcal{A}, P)\) be a measure space.

Definition: a filtration is a sequence \(\{\mathcal{A}_n|n \in \mathbb{N}\}\) of \(\sigma\)-algebras such that \(\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}\).

Definition: The SP \(\{Y_n|n \in \mathbb{N}\}\) is adapted to the filtration \(\{\mathcal{A}_n|n \in \mathbb{N}\}\) ⇔ \(Y_n\) is \(\mathcal{A}_n\)-measurable for all \(n\).

Intuition: growing information \(\mathcal{A}_n\)... And the value of \(Y_n\) is known as soon as the information \(\mathcal{A}_n\) is available.

The SP \(\{Y_n|n \in \mathbb{N}\}\) is a martingale w.r.t. the filtration \(\{\mathcal{A}_n|n \in \mathbb{N}\}\) ⇔

- (i) \(\{Y_n|n \in \mathbb{N}\}\) is adapted to the filtration \(\{\mathcal{A}_n|n \in \mathbb{N}\}\).
- (ii) \(E[|Y_n|] < \infty\) for all \(n\).
- (iii) \(E[Y_{n+1}|\mathcal{A}_n] = Y_n\) a.s. for all \(n\).
Definitions and basic comments

Remarks:

- (iii) shows that a martingale can be thought of as the fortune of a gambler betting on a fair game.

- (iii) $\Rightarrow E[Y_n] = E[Y_0]$ for all $n$ (mean-stationarity). Using (iii), we also have (for $k = 2, 3, \ldots$)

$$E[Y_{n+k}|\mathcal{A}_n] = \left(E\left[E[Y_{n+k}|\mathcal{A}_{n+k-1}]|\mathcal{A}_n\right]\right) = E[Y_{n+k-1}|\mathcal{A}_n] = \ldots = E[Y_{n+1}|\mathcal{A}_n] = Y_n$$

a.s. for all $n$. 
Let $\sigma(X_1, \ldots, X_n)$ be the smallest $\sigma$-algebra containing

$$\{X_i^{-1}(B) \mid B \in \mathcal{B}, \, i = 1, \ldots, n\}.$$ 

The SP $\{Y_n \mid n \in \mathbb{N}\}$ is a martingale w.r.t. the SP $\{X_n \mid n \in \mathbb{N}\} \iff \{Y_n \mid n \in \mathbb{N}\}$ is a martingale w.r.t. the filtration $\{\sigma(X_1, \ldots, X_n) \mid n \in \mathbb{N}\} \iff$

- (i) $Y_n$ is $\sigma(X_1, \ldots, X_n)$-measurable for all $n$.
- (ii) $\mathbb{E}[\mid Y_n \mid] < \infty$ for all $n$.
- (iii) $\mathbb{E}[Y_{n+1} \mid X_1, \ldots, X_n] = Y_n$ a.s. for all $n$.

Remark:
- (i) just states that "$Y_n$ is a function of $X_1, \ldots, X_n$ only".
Definitions and basic comments

A lot of examples...
Example 1

Let $X_1, X_2, \ldots$ be $\perp \perp$ integrable r.v.'s, with common mean 0. Let $Y_n := \sum_{i=1}^{n} X_i$. Then $\{Y_n\}$ is a martingale w.r.t. $\{X_n\}$.

Indeed,

- (i) is trivial.
- (ii) is trivial.
- (iii): with $A_n := \sigma(X_1, \ldots, X_n)$, we have
  
  \[
  E[Y_{n+1} | A_n] = E[Y_n + X_{n+1} | A_n] = E[Y_n | A_n] + E[X_{n+1} | A_n]
  \]
  
  \[
  = Y_n + E[X_{n+1}] = Y_n + 0 = Y_n \text{ a.s. for all } n,
  \]

  where we used that $Y_n$ is $A_n$-measurable and that $X_{n+1} \perp \perp A_n$.

Similarly, if $X_1, X_2, \ldots$ are $\perp \perp$ and integrable with means $\mu_1, \mu_2, \ldots$, respectively, $\{\sum_{i=1}^{n} (X_i - \mu_i)\}$ is a martingale w.r.t. $\{X_n\}$.
Example 2

Let $X_1, X_2, \ldots$ be $\perp \perp$ integrable r.v.'s, with common mean 1. Let $Y_n := \prod_{i=1}^{n} X_i$. Then \{$Y_n$\} is a martingale w.r.t. \{\$X_n$\}.

Indeed,

- (i) is trivial.
- (ii) is trivial.
- (iii): with $\mathcal{A}_n := \sigma(X_1, \ldots, X_n)$, we have

\[
E[Y_{n+1} | \mathcal{A}_n] = E[Y_nX_{n+1} | \mathcal{A}_n] = Y_nE[X_{n+1} | \mathcal{A}_n] = Y_nE[X_{n+1}] = Y_n
\]
a.s. for all $n$, where we used that $Y_n$ is $\mathcal{A}_n$-measurable (and hence behaves as a constant in $E[ . | \mathcal{A}_n]$).

Similarly, if $X_1, X_2, \ldots$ are $\perp \perp$ and integrable with means $\mu_1, \mu_2, \ldots$, respectively, \{$\prod_{i=1}^{n}(X_i/\mu_i)$\} is a martingale w.r.t. \{\$X_n$\}. 
Example 2

The previous example is related to basic models for stock prices.

Assume $X_1, X_2, \ldots$ are positive, $\perp$, and integrable r.v.’s. Let $Y_n := c \prod_{i=1}^{n} X_i$, where $c$ is the initial price.

The quantity $X_i - 1$ is the change in the value of the stock over a fixed time interval (one day, say) as a fraction of its current value. This multiplicative model (i) ensures nonnegativeness of the price and (ii) is compatible with the fact fluctuations in the value of a stock are roughly proportional to its price.

Various models are obtained for various distributions of the $X_i$’s:

- **Discrete Black-Scholes model**: $X_i = e^{\eta_i}$, where $\eta_i \sim \mathcal{N}(\mu, \sigma^2)$ for all $i$.
- **Binomial model**: $X_i = e^{-r}(1 + a)^{2\eta_i-1}$, where $\eta_i \sim \text{Bin}(1, p)$ for all $i$ ($r =$ interest rate, by which one discounts future rewards).
Example 3: random walks

Let $X_1, X_2, \ldots$ be i.i.d., with $P[X_i = 1] = p$ and $P[X_i = -1] = q = 1 - p$.

Let $Y_n := \sum_{i=1}^{n} X_i$.

The SP $\{Y_n\}$ is called a random walk.

Remarks:

- If $p = q$, the RW is said to be symmetric.
- Of course, from Example 1, we know that $\{(\sum_{i=1}^{n} X_i) - n(p - q)\}$ is a martingale w.r.t. $\{X_n\}$.

But other martingales exist for RWs...
Example 3: random walks

Consider the non-symmetric case \( (p \neq q) \) and let \( S_n := \left( \frac{q}{p} \right)^{Y_n} \).

Then \( \{S_n\} \) is a martingale w.r.t. \( \{X_n\} \).

Indeed,

- (i) is trivial.
- (ii): \( |S_n| \leq \max((q/p)^n, (q/p)^{-n}) \). Hence, \( E[|S_n|] < \infty \).
- (iii): with \( A_n := \sigma(X_1, \ldots, X_n) \), we have
  \[
  E[S_{n+1} | A_n] = E[(q/p)^{Y_n} (q/p)^{X_{n+1}} | A_n] = (q/p)^{Y_n} E[(q/p)^{X_{n+1}} | A_n] \\
  = S_n E[(q/p)^{X_{n+1}}] = S_n \left( (q/p)^1 \times p + (q/p)^{-1} \times q \right) = S_n, \text{ a.s. for all} \\
  \]
  where we used that \( Y_n \) is \( A_n \)-measurable and that \( X_{n+1} \perp \perp A_n \).
Example 3: random walks

Consider the symmetric case \((p = q)\) and let \(S_n := Y_n^2 - n\). Then \(\{S_n\}\) is a martingale w.r.t. \(\{X_n\}\).

Indeed,

- (i) is trivial.
- (ii): \(|S_n| \leq n^2 - n\). Hence, \(\mathbb{E}[|S_n|] < \infty\).
- (iii): with \(\mathcal{A}_n := \sigma(X_1, \ldots, X_n)\), we have

\[
\mathbb{E}[S_{n+1}|\mathcal{A}_n] = \mathbb{E}[(Y_n + X_{n+1})^2 - (n + 1)|\mathcal{A}_n]
\]

\[
= \mathbb{E}[(Y_n^2 + X_{n+1}^2 + 2Y_nX_{n+1}) - (n + 1)|\mathcal{A}_n]
\]

\[
= Y_n^2 + \mathbb{E}[X_{n+1}^2|\mathcal{A}_n] + 2Y_n\mathbb{E}[X_{n+1}|\mathcal{A}_n] - (n + 1)
\]

\[
= Y_n^2 + \mathbb{E}[X_{n+1}^2] + 2Y_n\mathbb{E}[X_{n+1}] - (n + 1)
\]

\[
= S_n \text{ a.s. for all } n,
\]

where we used that \(Y_n\) is \(\mathcal{A}_n\)-measurable and that \(X_{n+1} \perp \mathcal{A}_n\).
Example 4: De Moivre’s martingales

Let $X_1, X_2, \ldots$ be i.i.d., with $P[X_i = 1] = p$ and $P[X_i = -1] = q = 1 - p$.

Let $Y_0 := k \in \{1, 2, \ldots, m - 1\}$ be the initial state.

Let $Y_{n+1} := (Y_n + X_{n+1}) I[Y_n \notin \{0, m\}] + Y_n I[Y_n \in \{0, m\}]$.

\[\leadsto\text{ The SP } \{Y_n\} \text{ is called a random walk with absorbing barriers.}\]

Remarks:

\[\text{▷ Before being caught either in 0 or } m, \text{ this is just a RW.}\]

\[\text{▷ As soon as you get in 0 or } m, \text{ you stay there forever.}\]
Example 4: De Moivre’s martingales

Let $X_1, X_2, \ldots$ be i.i.d., with $P[X_i = 1] = p$ and $P[X_i = -1] = q = 1 - p$.

Let $Y_0 := k \in \{1, 2, \ldots, m - 1\}$ be the initial state.

Let $Y_{n+1} := (Y_n + X_{n+1}) I[Y_n \notin \{0, m\}] + Y_n I[Y_n \in \{0, m\}]$.

In this new setup and with this new definition of $Y_n$,

- In the non-symmetric case,
  \[
  \{S_n := \left(\frac{q}{p}\right)^{Y_n}\} \text{ is still a martingale w.r.t. } \{X_n\}.
  \]

- In the symmetric case,
  \[
  \{S_n := Y_n^2 - n\} \text{ is still a martingale w.r.t. } \{X_n\}.
  \]

(exercise).
Example 5: branching processes

Consider some population, in which each individual $i$ of the $Z_n$ individuals in the $n$th generation gives birth to $X_{n+1,i}$ children (the $X_{n,i}$'s are i.i.d., take values in $\mathbb{N}$, and have common mean $\mu < \infty$).

Assume that $Z_0 = 1$.

Then $\{Z_n/\mu^n\}$ is a martingale w.r.t. $\{\mathcal{A}_n := \sigma(\text{the } X_{m,i}\text{'s, } m \leq n)\}$.

Indeed,

\begin{itemize}
  \item (i), (ii): exercise...
  \item (iii): $E\left[\frac{Z_{n+1}}{\mu^{n+1}} | \mathcal{A}_n\right] = \frac{1}{\mu^{n+1}} E\left[\sum_{i=1}^{Z_n} X_{n+1,i} | \mathcal{A}_n\right] = \frac{1}{\mu^{n+1}} \sum_{i=1}^{Z_n} E[X_{n+1,i} | \mathcal{A}_n] = \frac{Z_0}{\mu^n}$ a.s. for all $n$.
\end{itemize}

In particular, $E\left[\frac{Z_n}{\mu^n}\right] = E\left[\frac{Z_0}{\mu^0}\right] = 1$. Hence, $E[Z_n] = \mu^n$ for all $n$. 
Consider an urn containing $b$ blue balls and $r$ red ones. Pick randomly some ball in the urn and put it back in the urn with an extra ball of the same color. Repeat this procedure.

This is a so-called contamination process.

Let $X_n$ be the number of red balls in the urn after $n$ steps. Let

$$R_n = \frac{X_n}{b + r + n}$$

be the proportion of red balls in the urn after $n$ steps.

Then $\{R_n\}$ is a martingale w.r.t. $\{X_n\}$.
Example 6: Polya’s urn

Indeed,

- (i) is trivial.
- (ii): $0 \leq |R_n| \leq 1$. Hence, $E[|R_n|] < \infty$.
- (iii): with $A_n := \sigma(X_1, \ldots, X_n)$, we have

\[
E[X_{n+1}|A_n] = (X_n + 1) \frac{X_n}{r + b + n} + (X_n + 0) \left(1 - \frac{X_n}{r + b + n}\right)
\]

\[
= (X_n + 1)X_n + X_n((r + b + n) - X_n)
\]

\[
= \frac{r + b + n + 1)X_n}{r + b + n}
\]

= $(r + b + n + 1)R_n$ a.s. for all $n$,

so that $E[R_{n+1}|A_n] = R_n$ a.s. for all $n$. 
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   3.1. Definitions and examples.
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Let \( (Y_n) \) be a martingale w.r.t. \( (A_n) \)

Let \( T : (\Omega, A, \mathbb{P}) \rightarrow \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\} \) be a r.v.

Definition: \( T \) is a stopping time w.r.t. \( (A_n) \) ⇔
\( (i) \) \( T \) is a.s. finite (i.e., \( \mathbb{P}[T < \infty] = 1 \)).
\( (ii) \) \([T = n] \in A_n \) for all \( n \).

Remarks:

- (ii) is the crucial assumption:
  it says that one knows, at time \( n \), on the basis of the “information" \( A_n \), whether \( T = n \) or not, that is, whether one should stop at \( n \) or not.

- (i) just makes (almost) sure that one will stop at some point.
Stopping times (examples)

(Kind of) examples...

Let \((Y_n)\) be a martingale w.r.t. \((\mathcal{A}_n)\). Let \(B \in \mathcal{B}\).

(A) Let \(T := \inf\{n \in \mathbb{N} \mid Y_n \in B\}\) be the time of 1st entry of \((Y_n)\) into \(B\). Then,

\[ [T = n] = [Y_0 \notin B, Y_1 \notin B, \ldots, Y_{n-1} \notin B, Y_n \in B] \in \mathcal{A}_n. \]

Hence, provided that \(T\) is a.s. finite, \(T\) is a ST.

(B) Let \(T := \sup\{n \in \mathbb{N} \mid Y_n \in B\}\) be the time of last escape of \((Y_n)\) out of \(B\). Then,

\[ [T = n] = [Y_n \in B, Y_{n+1} \notin B, Y_{n+2} \notin B, \ldots] \notin \mathcal{A}_n. \]

Hence, \(T\) is not a ST.
(C) Let $T := k$ a.s. (for some fixed integer $k$). Then, of course, (i) $T < \infty$ a.s. and (ii)

$$[T = n] = \begin{cases} \emptyset & \text{if } n \neq k \\ \Omega & \text{if } n = k, \end{cases}$$

which is in $\mathcal{A}_n$ for all $n$. Hence, $T$ is a ST.
Stopping times (properties)

Properties:

- \( [T = n] \in \mathcal{A}_n \ \forall n \) \( \overset{(1)}{\iff} \) \( [T \leq n] \in \mathcal{A}_n \ \forall n \) \( \overset{(2)}{\iff} \) \( [T > n] \in \mathcal{A}_n \ \forall n \).

  Indeed,

  \( \overset{(1)}{\Rightarrow} \) follows from \( [T \leq n] = \bigcup_{k=1}^{n} [T = k] \).

  \( \overset{(1)}{\Leftarrow} \) follows from \( [T = n] = [T \leq n] \ \backslash [T \leq n - 1] \).

  \( \overset{(2)}{\iff} \) follows from \( [T \leq n] = \Omega \ \backslash [T > n] \).

- \( T_1, T_2 \) are ST \( \Rightarrow \) \( T_1 + T_2, \text{max}(T_1, T_2), \text{and} \ \min(T_1, T_2) \) are ST (exercise).

- Let \((Y_n)\) be a martingale w.r.t. \((\mathcal{A}_n)\).

  Let \( T \) be a ST w.r.t. \((\mathcal{A}_n)\).

  Then \( Y_T := \sum_{n=0}^{\infty} Y_n \mathbb{1}_{[T=n]} \) is a r.v.

  Indeed, \([Y_T \in B] = \bigcup_{n=0}^{\infty} \{ [T = n] \ \cap [Y_n \in B] \} \in \mathcal{A} \).
A key lemma:

**Lemma:** Let \((Y_n)\) be a martingale w.r.t. \((\mathcal{A}_n)\). Let \(T\) be a ST w.r.t. \((\mathcal{A}_n)\). Then \(\{Z_n := Y_{\min(n,T)}\}\) is a martingale w.r.t. \((\mathcal{A}_n)\).

Proof: note that

\[
Z_n = Y_{\min(n,T)} = \sum_{k=0}^{n-1} Y_k \mathbb{I}[T=k] + Y_n \mathbb{I}[T \geq n].
\]

So

- (i): \(Z_n\) is \(\mathcal{A}_n\)-measurable for all \(n\).
- (ii): \(|Z_n| \leq \sum_{k=0}^{n-1} |Y_k| \mathbb{I}[T=k] + |Y_n| \mathbb{I}[T \geq n] \leq \sum_{k=0}^{n} |Y_k|\).

Hence, \(\mathbb{E}[|Z_n|] < \infty\).
(iii): we have

\[
\mathbb{E}[Z_{n+1} | \mathcal{A}_n] - Z_n = \mathbb{E}[Z_{n+1} - Z_n | \mathcal{A}_n] \\
= \mathbb{E}[(Y_{n+1} - Y_n) \mathbb{I}_{[T \geq n+1]} | \mathcal{A}_n] \\
= \mathbb{E}[(Y_{n+1} - Y_n) \mathbb{I}_{[T > n]} | \mathcal{A}_n] \\
= \mathbb{I}_{[T > n]} \mathbb{E}[Y_{n+1} - Y_n | \mathcal{A}_n] \\
= \mathbb{I}_{[T > n]} (\mathbb{E}[Y_{n+1} | \mathcal{A}_n] - Y_n) \\
= 0 \text{ a.s. for all } n,
\]

where we used that \( \mathbb{I}_{[T > n]} \) is \( \mathcal{A}_n \)-measurable. \( \square \)
**Corollary:** Let \((Y_n)\) be a martingale w.r.t. \((\mathcal{A}_n)\). Let \(T\) be a ST w.r.t. \((\mathcal{A}_n)\). Then \(\mathbb{E}[Y_{\min(n,T)}] = \mathbb{E}[Y_0]\) for all \(n\).

Proof: the lemma and the mean-stationarity of martingales yield \(\mathbb{E}[Y_{\min(n,T)}] = \mathbb{E}[Z_n] = \mathbb{E}[Z_0] = \mathbb{E}[Y_{\min(0,T)}] = \mathbb{E}[Y_0]\) for all \(n\). □

In particular, if the ST is such that \(T \leq k\) a.s. for some \(k\), we have that, for \(n \geq k\),

\[Y_{\min(n,T)} = Y_T \text{ a.s.},\]

so that

\[\mathbb{E}[Y_T] = \mathbb{E}[Y_0].\]
Stopped martingale

\[ \mathbb{E}[Y_T] = \mathbb{E}[Y_0] \] does not always hold.

Example: the doubling strategy, for which the winnings are

\[ Y_n = \sum_{i=1}^{n} C_i X_i, \]

where the \( X_i \)'s are i.i.d. \( \mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2} \) and \( C_i = 2^{i-1} b. \)

The SP \( (Y_n) \) is a martingale w.r.t. \( (X_n) \) (exercise).

Let \( T = \inf\{n \in \mathbb{N} | X_n = 1\} \) (exercise: \( T \) is a ST).

As we have seen, \( Y_T = b \) a.s., so that \( \mathbb{E}[Y_T] = b \neq 0 = \mathbb{E}[Y_0]. \)

However, as shown by the following result, \( \mathbb{E}[Y_T] = \mathbb{E}[Y_0] \) holds under much broader conditions than "\( T \leq k \) a.s."
Optional stopping theorem

**Theorem:** Let \((Y_n)\) be a martingale w.r.t. \((A_n)\). Let \(T\) be a ST w.r.t. \((A_n)\). Then if (i) \(\mathbb{E}[|Y_T|] < \infty\) and (ii) 
\[
\lim_{n \to \infty} \mathbb{E}[Y_n \mathbb{I}_{[T > n]}] = 0,
\]
we have \(\mathbb{E}[Y_T] = \mathbb{E}[Y_0]\).

**Proof:** since \(Y_{\min(n,T)} = Y_n \mathbb{I}_{[T > n]} + Y_T \mathbb{I}_{[T \leq n]}\), we have
\[
Y_T = Y_T \mathbb{I}_{[T \leq n]} + Y_T \mathbb{I}_{[T > n]} = (Y_{\min(n,T)} - Y_n \mathbb{I}_{[T > n]}) + Y_T \mathbb{I}_{[T > n]}.
\]
Taking expectations, we obtain
\[
\mathbb{E}[Y_T] = \mathbb{E}[Y_0] - \mathbb{E}[Y_n \mathbb{I}_{[T > n]}] + \mathbb{E}[Y_T \mathbb{I}_{[T > n]}].
\]
By taking the limit as \(n \to \infty\) and using (ii),
\[
\mathbb{E}[Y_T] = \mathbb{E}[Y_0] + \lim_{n \to \infty} \mathbb{E}[Y_T \mathbb{I}_{[T > n]}].
\]
The result follows from \(\lim_{n \to \infty} \mathbb{P}[T > n] = \mathbb{P}[T = \infty] = 0\). \(\square\)
Optional stopping theorem

**Theorem:** Let \((Y_n)\) be a martingale w.r.t. \((\mathcal{A}_n)\). Let \(T\) be a ST w.r.t. \((\mathcal{A}_n)\). Then if (i) \(\mathbb{E}[|Y_T|] < \infty\) and (ii) \(\lim_{n \to \infty} \mathbb{E}[Y_n \mathbb{I}_{[T > n]}] = 0\), we have \(\mathbb{E}[Y_T] = \mathbb{E}[Y_0]\).

**Particular sufficient conditions for (i), (ii):**

- (a) \(T \leq k\) a.s. Indeed,
  
  (i) \(\mathbb{E}[|Y_T|] = \mathbb{E}[|\sum_{n=0}^{k} Y_n \mathbb{I}_{[T=n]}|] \leq \sum_{n=0}^{k} \mathbb{E}[|Y_n|] < \infty\).
  
  (ii) \(Y_n \mathbb{I}_{[T > n]} = 0\) a.s. for \(n > k\). Hence, \(\mathbb{E}[Y_n \mathbb{I}_{[T > n]}] = 0\) for \(n > k\), so that (ii) holds.

- (b) \((Y_n)\) is uniformly integrable.
Optional stopping theorem (examples)

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$.

Let $Y_0 := k \in \{1, 2, \ldots, m - 1\}$ be the initial state.

Let $Y_{n+1} := (Y_n + X_{n+1}) \mathbb{I}_{[Y_n \not\in \{0, m\}]} + Y_n \mathbb{I}_{[Y_n \in \{0, m\}]}$.

$\leadsto$ The SP $(Y_n)$ is called a random walk with absorbing barriers.

In the symmetric case, $(Y_n)$ is a martingale w.r.t. $(X_n)$ (exercise).

Let $T := \inf\{n \in \mathbb{N} | Y_n \in \{0, m\}\}$ (exercise: $T$ is a stopping time, and the assumptions of the optional stopping thm are satisfied).

$\leadsto \mathbb{E}[Y_T] = \mathbb{E}[Y_0]$. 
Optional stopping theorem (examples)

Let \( p_k := \mathbb{P}[Y_T = 0] \).

Then
\[
\mathbb{E}[Y_T] = 0 \times p_k + m \times (1 - p_k)
\]
and
\[
\mathbb{E}[Y_0] = \mathbb{E}[k] = k,
\]
so that \( \mathbb{E}[Y_T] = \mathbb{E}[Y_0] \) yields
\[
m(1 - p_k) = k,
\]
that is, solving for \( p_k \),
\[
p_k = \frac{m - k}{m}.
\]
Optional stopping theorem (examples)

Is there a way to get $\mathbb{E}[T]$ (still in the symmetric case)?

We know that $(S_n := Y_n^2 - n)$ is also a martingale w.r.t. $(X_n)$ (exercise: with this martingale and the same ST, the assumptions of the optional stopping theorem are still satisfied).

$\sim \mathbb{E}[S_T] = \mathbb{E}[S_0], \text{where}$

\[ \mathbb{E}[S_T] = \mathbb{E}[Y_T^2] - \mathbb{E}[T] = \left(0^2 \times p_k + m^2 \times (1 - p_k)\right) - \mathbb{E}[T] \]

and

\[ \mathbb{E}[S_0] = \mathbb{E}[Y_0^2 - 0] = \mathbb{E}[k^2] = k^2. \]

Hence,

\[ \mathbb{E}[T] = m^2(1 - p_k) - k^2 = m^2 \times \frac{k}{m} - k^2 = k(m - k). \]
Optional stopping theorem (examples)

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$.
Let $Y_0 := k \in \{1, 2, \ldots, m - 1\}$ be the initial state.
Let $Y_{n+1} := (Y_n + X_{n+1}) \mathbb{I}[Y_n \notin \{0, m\}] + Y_n \mathbb{I}[Y_n \in \{0, m\}]$.

$\sim$ The SP $(Y_n)$ is called a random walk with absorbing barriers.

In the non-symmetric case, $\left(S_n := \left(\frac{q}{p}\right)^{Y_n}\right)$ is a martingale w.r.t. $(X_n)$. Let $T := \inf\{n \in \mathbb{N}|S_n \in \{0, m\}\}$ (exercise: $T$ is a stopping time, and the assumptions of the optional stopping thm are satisfied). $\sim \mathbb{E}[S_T] = \mathbb{E}[S_0]$. 
Let again $p_k := \mathbb{P}[Y_T = 0]$. Then, since

$$\mathbb{E}[S_T] = \mathbb{E} \left[ \left( \frac{q}{p} \right)^{Y_T} \right] = \left( \frac{q}{p} \right)^0 \times p_k + \left( \frac{q}{p} \right)^m (1 - p_k),$$

and

$$\mathbb{E}[S_0] = \mathbb{E} \left[ \left( \frac{q}{p} \right)^{Y_0} \right] = \mathbb{E} \left[ \left( \frac{q}{p} \right)^k \right] = \left( \frac{q}{p} \right)^k,$$

we deduce that

$$\left( \frac{q}{p} \right)^0 \times p_k + \left( \frac{q}{p} \right)^m (1 - p_k) = \left( \frac{q}{p} \right)^k,$$

that is, solving for $p_k$,

$$p_k = \frac{\left( \frac{q}{p} \right)^k - \left( \frac{q}{p} \right)^m}{1 - \left( \frac{q}{p} \right)^m}.$$
Optional stopping theorem (examples)

Is there a way to get \( \mathbb{E}[T] \) here as well?

In the non-symmetric case, \( (R_n := Y_n - \min(n, T)(p - q)) \) is a martingale w.r.t. \( (X_n) \) (exercise: check this, and check that the optional stopping thm applies with \( (R_n) \) and \( T \)).

\[
\sim \quad \mathbb{E}[R_T] = \mathbb{E}[R_0], \quad \text{where}
\]

\[
\mathbb{E}[R_T] = \mathbb{E}[Y_T - T(p-q)] = \mathbb{E}[Y_T] - (p-q)\mathbb{E}[T] = \left(0 \times p_k + m \times (1-p_k)\right) \quad \text{and}
\]

\[
\mathbb{E}[R_0] = \mathbb{E}[Y_0 - \min(0, T)(p-q)] = \mathbb{E}[Y_0] = \mathbb{E}[k] = k.
\]

Hence,

\[
\mathbb{E}[T] = \frac{m(1-p_k) - k}{p-q} = \frac{m\left(1 - \left(\frac{q}{p}\right)^k\right) - k\left(1 - \left(\frac{q}{p}\right)^m\right)}{(p-q)\left(1 - \left(\frac{q}{p}\right)^m\right)}.
\]
Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
   3.1. Definitions and examples.
   3.2. Stopping times and the optional stopping theorem.
   3.3. Sub- and super- martingales, Limiting results.
Sub- and super-martingales

Not every game is fair...
There are also favourable and defavourable games.

Therefore, we introduce the following concepts:

The SP \((Y_n)_{n \in \mathbb{N}}\) is a submartingale w.r.t. the filtration \((\mathcal{A}_n)_{n \in \mathbb{N}}\) ⇔

- (i) \((Y_n)_{n \in \mathbb{N}}\) is adapted to the filtration \((\mathcal{A}_n)_{n \in \mathbb{N}}\).
- (ii)' \(\mathbb{E}[Y_n^+] < \infty\) for all \(n\).
- (iii)' \(\mathbb{E}[Y_{n+1} | \mathcal{A}_n] \geq Y_n\) a.s. for all \(n\).

The SP \((Y_n)_{n \in \mathbb{N}}\) is a supermartingale w.r.t. the filtration \((\mathcal{A}_n)_{n \in \mathbb{N}}\) ⇔

- (i) \((Y_n)_{n \in \mathbb{N}}\) is adapted to the filtration \((\mathcal{A}_n)_{n \in \mathbb{N}}\).
- (ii)" \(\mathbb{E}[Y_n^-] < \infty\) for all \(n\).
- (iii)" \(\mathbb{E}[Y_{n+1} | \mathcal{A}_n] \leq Y_n\) a.s. for all \(n\).
Sub- and super-martingales

Remarks:

- (iii)' shows that a submartingale can be thought of as the fortune of a gambler betting on a favourable game.
  
  \[(iii)' \implies \mathbb{E}[Y_n] \geq \mathbb{E}[Y_0] \text{ for all } n.\]

- (iii)” shows that a supermartingale can be thought of as the fortune of a gambler betting on a defavourable game.
  
  \[(iii)” \implies \mathbb{E}[Y_n] \leq \mathbb{E}[Y_0] \text{ for all } n.\]

- \((Y_n)\) is a submartingale w.r.t. \((\mathcal{A}_n)\)
  \[\iff (−Y_n) \text{ is a supermartingale w.r.t. } (\mathcal{A}_n).\]

- \((Y_n)\) is a martingale w.r.t. \((\mathcal{A}_n)\)
  \[\iff (Y_n) \text{ is both a sub- and a supermartingale w.r.t. } (\mathcal{A}_n).\]
Consider the following strategy for the fair version of roulette (without the "0" slot):

Bet $b$ euros on an even result. If you win, stop. If you lose, bet $2b$ euros on an even result. If you win, stop. If you lose, bet $4b$ euros on an even result. If you win, stop... And so on...

How good is this strategy?

(a) If you first win in the $n$th game, your total winning is

$$-\sum_{i=0}^{n-2} 2^i b + 2^{n-1} b = b$$

⇒ Whatever the value of $n$ is, you win $b$ euros with this strategy.
(b) You will a.s. win. Indeed, let $T$ be the time index of first success. Then

$$\mathbb{P}[T < \infty] = \sum_{n=1}^{\infty} \mathbb{P}[n - 1 \text{ first results are "odd", then "even"}].$$

But

\[
\begin{align*}
\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{n-1} \frac{1}{2} &= \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{n} = 1.
\end{align*}
\]

(c) The expected amount you lose just before you win is

\[
0 \times \frac{1}{2} + b \times \left( \frac{1}{2} \right)^{2} + (b+2b) \times \left( \frac{1}{2} \right)^{3} + \ldots + \left( \sum_{i=0}^{n-2} 2^{i} b \right) \left( \frac{1}{2} \right)^{n} + \ldots = \infty
\]

$\Rightarrow$ Your expected loss is infinite!

(d) You need an unbounded wallet...
Let us try to formalize strategies...

Consider the SP $(X_n)_{n \in \mathbb{N}}$, where $X_n$ is your winning per unit stake in game $n$. Denote by $(A_n)_{n \in \mathbb{N}}$ the corresponding filtration $(A_n = \sigma(X_1, \ldots, X_n))$.

Definition: A gambling strategy (w.r.t. $(X_n)$) is a SP $(C_n)_{n \in \mathbb{N}}$ such that $C_n$ is $A_{n-1}$-measurable for all $n$.

Remarks:
- $C_n = C_n(X_1, \ldots, X_{n-1})$ is what you will bet in game $n$.
- $A_0 = \{\emptyset, \Omega\}$. 
Sub- and super-martingales

Using some strategy \((C_n)\), your total winning after \(n\) games is

\[
Y_n^{(C)} = \sum_{i=1}^{n} C_i X_i.
\]

A natural question:
Is there any way to choose \((C_n)\) so that \((Y_n^{(C)})\) is "nice"?

Consider the "blind" strategy \(C_n = 1\) (for all \(n\)), that consists in betting 1 euro in each game, and denote by \((Y_n = \sum_{i=1}^{n} X_i)\) the corresponding process of winnings.

Then, here is the answer:

**Theorem**: Let \((C_n)\) be a gambling strategy with nonnegative and bounded r.v.'s. Then if \((Y_n)\) is a martingale, so is \((Y_n^{(C)})\). If \((Y_n)\) is a submart., so is \((Y_n^{(C)})\). And if \((Y_n)\) is a supermart., so is \((Y_n^{(C)})\).
Sub- and super-martingales

Proof:

- (i) is trivial.
- (ii): $|Y_n^{(C)}| \leq \sum_{i=1}^{n} a_i |X_i|$. Hence, $\mathbb{E}[|Y_n^{(C)}|] < \infty$.
- (iii),(iii)',(iii)”: with $\mathcal{A}_n := \sigma(X_1, \ldots, X_n)$, we have
  \[
  \mathbb{E}[Y_{n+1}^{(C)}|\mathcal{A}_n] = \mathbb{E}[Y_n^{(C)} + C_{n+1}X_{n+1}|\mathcal{A}_n]
  = Y_n^{(C)} + C_{n+1} \mathbb{E}[X_{n+1}|\mathcal{A}_n]
  = Y_n^{(C)} + C_{n+1} \mathbb{E}[Y_{n+1} - Y_n|\mathcal{A}_n]
  = Y_n^{(C)} + C_{n+1} \left( \mathbb{E}[Y_{n+1}|\mathcal{A}_n] - Y_n \right),
  \]
  where we used that $C_{n+1}$ is $\mathcal{A}_n$-measurable. Since $C_{n+1} \geq 0$, the result follows.

Remark: The second part was checked for martingales only.
Exercise: check (ii)' and (ii)"...
Convergence of martingales

**Theorem**: let \((Y_n)\) be a submartingale w.r.t. \((\mathcal{A}_n)\). Assume that, for some \(M\), \(\mathbb{E}[Y_n^+] \leq M\) for all \(n\). Then

(i) \(\exists Y_\infty\) such that \(Y_n \xrightarrow{a.s.} Y_\infty\) as \(n \to \infty\).

(ii) If \(\mathbb{E}[|Y_0|] < \infty\), \(\mathbb{E}[|Y_\infty|] < \infty\).

The following results directly follow:

**Corollary 1**: let \((Y_n)\) be a submartingale or a supermartingale w.r.t. \((\mathcal{A}_n)\). Assume that, for some \(M\), \(\mathbb{E}[|Y_n|] \leq M\) for all \(n\). Then \(\exists Y_\infty\) (satisfying \(\mathbb{E}[|Y_\infty|] < \infty\)) such that \(Y_n \xrightarrow{a.s.} Y_\infty\) as \(n \to \infty\).

**Corollary 2**: let \((Y_n)\) be a negative submartingale or a positive supermartingale w.r.t. \((\mathcal{A}_n)\). Then \(\exists Y_\infty\) such that \(Y_n \xrightarrow{a.s.} Y_\infty\) as \(n \to \infty\).
Example: products of r.v.’s

Let $X_1, X_2, \ldots$ be i.i.d. r.v.’s, with common distribution

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<th>distribution of $X_i$</th>
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<td>values</td>
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<td>probabilities</td>
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The $X_i$’s are integrable r.v.’s with common mean 1, so that $(Y_n = \prod_{i=1}^{n} X_i)$ is a (positive) martingale w.r.t. $(X_n)$ (example 2 in the previous lecture).

Consequently, $\exists Y_\infty$ such that $Y_n \xrightarrow{a.s.} Y_\infty$ as $n \to \infty$.

We showed, in Lecture #4, that $Y_n \xrightarrow{P} 0$ as $n \to \infty$ so that $Y_\infty = 0$ a.s.

But we also showed there that convergence in $L^1$ does not hold. To ensure $L^1$-convergence, one has to require uniform integrability of $(Y_n)$. 
Example: Polya’s urn

Consider an urn containing $b$ blue balls and $r$ red ones. Pick randomly some ball in the urn and put it back in the urn with an extra ball of the same color. Repeat this procedure.

Let $X_n$ be the number of red balls in the urn after $n$ steps. Let $R_n = \frac{X_n}{b + r + n}$ be the proportion of red balls after $n$ steps.

We know that $(R_n)$ is a martingale w.r.t. $(X_n)$.

Now, $|R_n| \leq 1 \Rightarrow \mathbb{E}[|R_n|] \leq 1$, so that $\exists R_\infty$ (satisfying $\mathbb{E}[|R_\infty|] < \infty$) such that $R_n \xrightarrow{a.s.} R_\infty$ as $n \to \infty$.

Clearly, uniform integrability holds. Hence, $R_n \xrightarrow{L^1} R_\infty$ as $n \to \infty$.

Remark: it can be shown that $R_\infty$ has a beta distribution:

$$
\mathbb{P}[R_\infty \leq u] = \binom{b + r}{r} \int_0^u x^{r-1}(1 - x)^{b-1} \, dx, \quad u \in (0, 1).
$$
Example: branching processes

Consider some population, in which each individual $i$ of the $Z_n$ individuals in the $n$th generation gives birth to $X_{n,i}$ children (the $X_{n,i}$’s are i.i.d., take values in $\mathbb{N}$, and have common mean $\mu < \infty$).

Assume that $Z_0 = 1$.

Then $(Y_n := Z_n/\mu^n)$ is a martingale w.r.t. $(\mathcal{A}_n := \sigma(X_{n,i}, n \text{ fixed}))$.

What can be said about the long-run state?

$\mathbb{E}[|Y_n|] = \mathbb{E}[Y_0] = 1$ for all $n \Rightarrow \exists Y_\infty$ (satisfying $\mathbb{E}[|Y_\infty|] < \infty$) such that $Y_n \overset{a.s.}{\rightarrow} Y_\infty$ as $n \rightarrow \infty$.

$\sim Z_n \approx Y_\infty \mu^n$ for large $n$. 
Example: branching processes

Case 1: \( \mu \leq 1 \).

If \( p_0 := \mathbb{P}[X_{n,i} = 0] > 0 \), \( Z_n \xrightarrow{a.s.} Z_\infty := 0 \) as \( n \to \infty \).

Proof:

Let \( k \in \mathbb{N}_0 \). Then

\[
\mathbb{P}[Z_\infty = k] = \mathbb{P}[Z_n = k \ \forall n \geq n_0] = \mathbb{P}[Z_{n_0} = k] \left( \mathbb{P}[Z_{n_0 + 1} = k | Z_{n_0} = k] \right)^\infty
\]

where

\[
\mathbb{P}[Z_{n_0 + 1} = k | Z_{n_0} = k] \leq \mathbb{P}[Z_{n_0 + 1} \neq 0 | Z_{n_0} = k] = 1 - (p_0)^k < 1.
\]

Therefore, \( \mathbb{P}[Z_\infty = k] = 0 \) for all \( k \in \mathbb{N}_0 \), so that \( Z_\infty = 0 \) a.s. \( \square \)
Example: branching processes

Case 2: \( \mu > 1 \).

Then \( Z_n \xrightarrow{a.s.} Z_\infty \) as \( n \to \infty \), where the distribution of \( Z_\infty \) is given by

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<td>probabilities</td>
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<td>( \pi )</td>
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<td>( 1 - \pi )</td>
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where \( \pi \in (0, 1) \) is the unique solution of \( g(s) = s \), where, defining \( p_k := \mathbb{P}[X_{n,i} = k] \), we let \( g(s) := \sum_{k=0}^{\infty} p_k s^k \).

Proof:

Clearly, since \( Z_n \approx Y_\infty \mu^n \) for large \( n \), \( Z_n \xrightarrow{a.s.} Z_\infty \) as \( n \to \infty \), where \( Z_\infty \) only assumes the values 0 and \( \infty \), with proba \( a \) and 1 – \( a \), respectively, say.

We may assume that \( p_0 > 0 \) (clearly, if \( p_0 = 0 \), \( Z_\infty = \infty \) a.s.)
Example: branching processes

There is indeed a unique \( \pi \in (0, 1) \) such that \( g(\pi) = \pi \), since \( g \) is monotone increasing, \( g(0) = \rho_0 > 0 \), and \( g'(1) = \mu > 1 \).

\( \sim (\pi Z_n) \) is a martingale w.r.t. \( (A_n) \), since

\[
\mathbb{E}[\pi Z_{n+1} | A_n] = \mathbb{E}[\pi \sum_{i=1}^{Z_n} X_{n+1,i} | A_n] = (\mathbb{E}[\pi X_{n+1,1} | A_n])^{Z_n} \\
= (\mathbb{E}[\pi X_{n+1,1}])^{Z_n} = \left( \sum_{k=0}^{\infty} \pi^k p_k \right)^{Z_n} = (g(\pi))^{Z_n} = \pi^{Z_n}
\]
a.s. for all \( n \).

\( (\pi Z_n) \) is also uniformly integrable, so that \( \pi Z_n \overset{L^1}{\to} \pi Z_\infty \) as \( n \to \infty \). Therefore, \( \mathbb{E}[\pi Z_\infty] = \pi^0 \times a + \pi^\infty \times (1 - a) = a \) is equal to \( \mathbb{E}[\pi Z_0] = \mathbb{E}[\pi^1] = \pi \).

Hence, \( \mathbb{P}[Z_\infty = 0] = \pi \) and \( \mathbb{P}[Z_\infty = \infty] = 1 - \pi \).  \( \square \)
Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
   4.1. Definitions and examples.
   4.2. Strong Markov property, number of visits.
   4.3. Classification of states.
   4.4. Computation of \( R \) and \( F \).
   4.5. Asymptotic behavior.
Markov chains

The importance of these processes comes from two facts:

- there is a large number of physical, biological, economic, and social phenomena that can be described in this way, and
- there is a well-developed theory that allows for doing the computations and obtaining explicit results...
Definitions and examples

Let $S$ be a finite or countable set (number its elements using $i = 1, 2, \ldots$).

Let $(X_n)_{n \in \mathbb{N}}$ be a SP with $X_n : (\Omega, \mathcal{A}, P) \to S$ for all $n$.

Definition: $(X_n)$ is a Markov chain (MC) on $S$ ⇔

$$
\mathbb{P}[X_{n+1} = j | X_0, X_1, \ldots, X_n] = \mathbb{P}[X_{n+1} = j | X_n] \quad \forall n \forall j.
$$

Remarks:

- The equation above is the so-called Markov property. It states that the future does only depend on the present state of the process, and not on its past.

- $S$ is the state space.

- The elements of $S$ are the states.
Definitions and examples

Definition: The MC \((X_n)\) is **homogeneous** \((\sim \text{HMC})\) \iff

\[ P[X_{n+1} = j | X_n = i] = P[X_1 = j | X_0 = i] \quad \forall n \forall i, j. \]

For a HMC, one can define the **transition probabilities**

\[ p_{ij} = P[X_{n+1} = j | X_n = i] \quad \forall i, j, \]

which are usually collected in the transition matrix \(P = (p_{ij})\).

The transition matrix \(P\) is a "stochastic matrix", which means that

- \(p_{ij} \in [0, 1]\) for all \(i, j\).
- \(\sum_j p_{ij} = 1\) for all \(i\).

In vector notation, \(P1 = 1\), where 1 stands for the vector of ones with the appropriate dimension.
Example 1: random walk

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$.

Let $Y_i := \sum_{i=1}^{n} X_i$ be the corresponding random walk.

$(Y_n)$ is a HMC on $S = \mathbb{Z}$ with transition matrix

$$P = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & q & 0 & p & \ddots \\ \ddots & q & 0 & p & 0 \\ \ddots & q & 0 & p & 0 \\ & \ddots & q & 0 & \ddots \end{pmatrix}.$$
Example 2: rw with absorbing barriers

Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = -1] = q = 1 - p$.

Let $Y_0 := k \in \{1, 2, \ldots, m - 1\}$ be the initial state.

Let $Y_{n+1} := (Y_n + X_{n+1}) \mathbb{1}[Y_n \notin \{0, m\}] + Y_n \mathbb{1}[Y_n \in \{0, m\}]$.

$(Y_n)$ is a HMC on $S = \{0, 1, \ldots, m\}$ with transition matrix

$$P = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
q & 0 & p & \cdots & 0 \\
q & 0 & 0 & \cdots & p \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
q & 0 & 0 & \cdots & 1
\end{pmatrix}.$$
Let $X_1, X_2, \ldots$ be i.i.d., with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = 0] = q = 1 - p$.

Let $Y_0 := 0$ be the initial state.

Let $Y_{n+1} := (Y_n + 1) \mathbb{I}[X_{n+1} = 1] + 0 \times \mathbb{I}[X_{n+1} = 0]$.

$\sim (Y_n)$ is a HMC on $S = \mathbb{N}$ with transition matrix

$$P = \begin{pmatrix}
q & p & 0 & 0 & \cdots & 0 \\
q & 0 & p & 0 & \cdots & 0 \\
q & 0 & 0 & p & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
q & 0 & 0 & 0 & \cdots & 0 \\
q & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.$$
Example 4: discrete queue models

Let $Y_n$ be the number of clients in a queue at time $n$ ($Y_0 = 0$). Let $X_n$ be the number of clients entering the shop between time $n - 1$ and $n$ ($X_n$ i.i.d., with $\mathbb{P}[X_n = i] = p_i; \sum_{i=0}^{\infty} p_i = 1$). Assume a service needs exactly one unit of time to be completed.

Then

$$Y_{n+1} = (Y_n + X_n - 1) \mathbb{I}[Y_n > 0] + X_n \mathbb{I}[Y_n = 0],$$

and $(Y_n)$ is a HMC on $S = \mathbb{N}$ with transition matrix

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ p_0 & p_1 & p_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ p_0 & p_1 & p_2 & \cdots \end{pmatrix}.$$
Example 5: stock management

Let $X_n$ be the number of units on hand at the end of day $n$ ($X_0 = M$).
Let $D_n$ be the demand on day $n$ ($D_n$ i.i.d., $\mathbb{P}[D_n = i] = p_i$; $\sum_{i=0}^{\infty} p_i = 1$).
Assume that if $X_n \leq m$, it is (instantaneously) set to $M$ again.

Then, letting $x^+ = \max(x, 0)$, we have

$$X_{n+1} = (X_n - D_{n+1})^+ \mathbb{I}[X_n > m] + (M - D_{n+1})^+ \mathbb{I}[X_n \leq m],$$

and $(X_n)$ is a HMC on $S = \{0, 1, \ldots, M\}$ (exercise: derive $P$).

Questions:

- if we make 12$ profit on each unit sold but it costs 2$ a day to store items, what is the long-run profit per day of this inventory policy?
- How to choose $(m, M)$ to maximize profit?
Example 6: income classes

Assume that from one generation to the next, families change their income group "Low", "Middle", or "High" (state 1,2, and 3, respectively) according to a HMC with transition matrix

\[ P = \begin{pmatrix} .6 & .3 & .1 \\ .1 & .7 & .1 \\ .1 & .3 & .6 \end{pmatrix}. \]

Questions:

- Do the fractions of the population in the three income classes stabilize as time goes on?
- If this happens, how can we compute the limiting proportions from \( P \)?
Higher-order transition probabilities

We let \( P = (p_{ij}) \), where \( p_{ij} = \mathbb{P}[X_1 = j | X_0 = i] \).
Now, define \( P^{(n)} = (p^{(n)}_{ij}) \), where \( p^{(n)}_{ij} = \mathbb{P}[X_n = j | X_0 = i] \).

What is the link between \( P \) and \( P^{(n)} \)?

\[ \blacktriangleright \textbf{Theorem}: P^{(n)} = P^n. \]

Proof: the result holds for \( n = 1 \). Now, assume it holds for \( n \).
Then \( (P^{(n+1)})_{ij} = \mathbb{P}[X_{n+1} = j | X_0 = i] = \sum_k \mathbb{P}[X_{n+1} = j, X_n = k | X_0 = i] = \sum_k \mathbb{P}[X_{n+1} = j | X_n = k, X_0 = i] \mathbb{P}[X_n = k | X_0 = i] = \sum_k (P^{(n)})_{ik} (P^{(1)})_{kj} = (P^{(n)} P)_{ij} = (P^n P)_{ij} = (P^{n+1})_{ij} \), so that the result holds for \( n + 1 \) as well. \[ \square \]
Of course, this implies that
\[ P^{(n+m)} = P^{n+m} = P^n P^m = P^{(n)} P^{(m)}, \]
that is,

\[ P[X_{n+m} = j | X_0 = i] = \sum_k P[X_n = k | X_0 = i] P[X_m = j | X_0 = k], \]

which are the so-called Chapman-Kolmogorov equations.
Higher-order transition probabilities

Clearly, the distribution of $X_n$ is of primary interest.

Let $a^{(n)}$ be the (line) vector with $j$th component 
$$(a^{(n)})_j = \mathbb{P}[X_n = j].$$

\(\iff\) \textbf{Theorem:} $a^{(n)} = a^{(0)} P^n$.

\textbf{Proof:} using the total probability formula, we obtain
$$(a^{(n)})_j = \mathbb{P}[X_n = j] = \sum_k \mathbb{P}[X_n = j \mid X_0 = k] \mathbb{P}[X_0 = k] = \sum_k (a^{(0)})_k (P^{(n)})_{kj} = (a^{(0)} P^{(n)})_j = (a^{(0)} P^n)_j,$$
which establishes the result.  \(\square\)

This shows that one can very easily compute the distribution of $X_n$ in terms of

- the distribution of $X_0$, and
- the transition matrix $P$.  

Higher-order transition probabilities

**Proposition:** let \((X_n)\) be a HMC on \(S\). Then

\[
P[X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n | X_0 = i_0],
\]

\[
P[X_{m+1} = i_1, X_{m+2} = i_2, \ldots, X_{m+n} = i_n | X_m = i_0],
\]

and

\[
P[X_{m+1} = i_1, X_{m+2} = i_2, \ldots, X_{m+n} = i_n | X_0 = j_0, X_1 = j_1, \ldots, X_m = i_0]
\]

all are equal to \(p_{i_0i_1}p_{i_1i_2} \cdots p_{i_{n-1}i_n}\).

Proof: exercise...