Stochastic Processes (Lecture #8)

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Outline of the course

1. A short introduction.
2. Basic probability review.
3. Martingales.
4. Markov chains.
5. Markov processes, Poisson processes.
6. Brownian motions.
    - 6.1. Definition.
    - 6.2. Markov property, martingales.
    - 6.3. The reflection principle.
    - 6.4. Gaussian processes and Brownian bridges.
    - 6.5. Stochastic calculus.

BM and Gaussian processes

Let (X_t) be a SP.

Definition: (X_t) is a Gaussian process $\Leftrightarrow$ for all k, for all $t_1 < t_2 < \ldots < t_k$, $(X_{t_1}, \ldots, X_{t_k})'$ is a Gaussian r.v.

Remark: the distribution of a Gaussian process is completely determined by

- its mean function $t \mapsto \mathbb{E}[X_t]$ and
- its autocovariance function $(s, t) \mapsto \mathrm{Cov}[X_s, X_t]$.

Proposition: A standard BM (X_t) is a Gaussian process with mean function $t \mapsto \mathbb{E}[X_t] = 0$ and autocovariance function $(s, t) \mapsto \mathrm{Cov}[X_s, X_t] = \min(s, t)$.

This might also be used as an alternative definition for BMs...

Proof: let (X_t) be a standard BM.

(i) For $s < t$, $X_t - X_s \stackrel{\mathcal{D}}{=} X_{t-s} - X_{s-s} = X_{t-s} \sim \mathcal{N}(0, t-s)$.

By using the independence between disjoint increments, we obtain, for $0 =: t_0 < t_1 < t_2 < \ldots < t_k$,

$$\begin{pmatrix} X_{t_1} - X_0 \\ X_{t_2} - X_{t_1} \\ \vdots \\ X_{t_k} - X_{t_{k-1}} \end{pmatrix} \sim \mathcal{N}(0, \Lambda),$$

where $\Lambda = (\lambda_{ij})$ is diagonal with $\lambda_{ii} = t_i - t_{i-1}$.

BM and Gaussian processes

Hence,

$$\sum_{i=1}^{k} v_i X_{t_i} = v' \begin{pmatrix} X_{t_1} \\ X_{t_2} \\ \vdots \\ X_{t_k} \end{pmatrix} = v' \begin{pmatrix} 1 & 0 & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} X_{t_1} - X_0 \\ X_{t_2} - X_{t_1} \\ \vdots \\ X_{t_k} - X_{t_{k-1}} \end{pmatrix}$$

is normally distributed, so that (X_t) is a Gaussian process.

(ii) Clearly, $t \mapsto \mathbb{E}[X_t] = 0$ for all t.

(iii) Eventually, assuming that $s < t$, we have

$$\mathrm{Cov}[X_s, X_t] = \mathrm{Cov}[X_s, X_s + (X_t - X_s)] = \mathrm{Var}[X_s] + \mathrm{Cov}[X_s, X_t - X_s] =$$

$$= s + \mathrm{Cov}[X_s - X_0, X_t - X_s] = s + 0 = \min(s, t).$$

$\square$

Brownian bridges

Let $(X_t)_{t\geq 0}$ be a BM.

Definition: if (X_t) is a BM, $(X_t - tX_1)_{0\leq t\leq 1}$ is a Brownian bridge.

Alternatively, it can be defined as a Gaussian process (over $(0,1)$) with mean function $t \mapsto \mathbb{E}[X_t] = 0$ and autocovariance function $(s,t) \mapsto \mathrm{Cov}[X_s, X_t] = \min(s,t)(1-\max(s,t))$ (exercise).

Application:

Let $X_1, \ldots, X_n$ be i.i.d. with cdf F.
Let $F_n(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{[X_i\leq x]}$ be the empirical cdf.

The LLN implies that $F_n(x) \stackrel{a.s.}{\to} \mathbb{E}[\mathbb{I}_{[X_1\leq x]}] = F(x)$ as $n\to\infty$.
Actually, it can be shown that $\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \stackrel{a.s.}{\to} 0$ as $n\to\infty$ (Glivenko-Cantelli theorem).

Brownian bridges

Assume that $X_1, \ldots, X_n$ are i.i.d. $\mathrm{Unif}(0,1)$ $(F(x) = x\mathbb{I}_{[x\in[0,1]]} + \mathbb{I}_{[x>1]})$.
Let $U_n(x) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\mathbb{I}_{[X_i \le x]} - x\right)$, $x \in [0,1]$.

Then it can be shown that, as $n \to \infty$,

$$\sup_{x\in[0,1]} |U_n(x)| \xrightarrow{\mathcal{D}} \sup_{x\in[0,1]} |U(x)|,$$

where $(U(x))_{0\le x\le 1}$ is a Brownian bridge (Donsker's theorem).
Coming back to the setup where $X_1, \ldots, X_n$ are i.i.d. with (unknown) cdf F, the result above allows for testing

$$\begin{cases} \mathcal{H}_0 : & F = F_0 \\ \mathcal{H}_1 : & F \neq F_0, \end{cases}$$

where F_0 is some fixed (continuous) cdf.

Brownian bridges

The so-called Kolmogorov-Smirnov test consists in rejecting $\mathcal{H}_0$ if the value of

$$\sup_{x\in[0,1]} |U_n(x)| := \sup_{x\in[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\mathbb{I}_{[F_0(X_i)\leq x]} - x \right) \right|$$

exceeds some critical value (that is computed from Donsker's theorem).
This is justified by the fact that, under $\mathcal{H}_0$, $F_0(X_1), \ldots, F_0(X_n)$ are i.i.d. $\mathrm{Unif}(0,1)$ (exercise).

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Stochastic calculus

Recall the definition of a (S)BM:

Definition: the SP $(X_t)_{t\geq 0}$ is a standard Brownian motion $\Leftrightarrow$

- $X_0 = 0$.
- for all $t > 0$, $X_t \sim \mathcal{N}(0, t)$;
- the increments in disjoint time-intervals are $\perp\!\!\!\perp$, i.e.

  $\forall 0 \leq t_1 < t_2 < \ldots < t_k, \quad X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_k} - X_{t_{k-1}}$ are $\perp\!\!\!\perp$;

- the increments in equal-length time-intervals are stationary, i.e.

$$\forall s, t > 0, \quad X_{t+s} - X_t \overset{\mathcal{D}}{=} X_s - X_0;$$

- the sample paths of $(X_t)_{t\geq 0}$ are a.s. continuous.

Stochastic integrals

In stochastic finance, it is crucial to be able to define integrals such as $\int_0^t C_x\, dB_x$, where $(C_x)_{x\geq 0}$ is some SP that is adapted to the SBM $(B_x)_{x\geq 0}$.

The first idea is to mimic (pathwise) the definition of

$$\int_0^t f(x)\, dg(x) = \lim_{\max(t_i - t_{i-1})\to 0} \sum_{i=1}^{n} f(y_i) \Big[g(t_i) - g(t_{i-1}) \Big],$$

where $0 = t_0 < t_1 < \ldots < t_n = t$ and y_i is an arbitrary point in $[t_{i-1}, t_i]$.

$\leadsto$ this leads to the temptative definition

$$\Big(\int_0^t C_x\, dB_x \Big)(\omega) = \lim_{\max(t_i - t_{i-1})\to 0} \sum_{i=1}^{n} C_{y_i}(\omega) \Big[B_{t_i}(\omega) - B_{t_{i-1}}(\omega) \Big].$$

However, as $t \mapsto B_t$ is (a.s.) nowhere differentiable, this limit does not exist for all (C_x).

Therefore, we will rather adopt a stochastic limit as a definition:

Definition:

$$\int_0^t C_x \, dB_x = L^2 - \lim_{\max(t_i - t_{i-1}) \to 0} \sum_{i=1}^{n} C_{t_{i-1}} \left[B_{t_i} - B_{t_{i-1}} \right].$$

Remarks:

- unlike in the pathwise definition, the result may depend on the point $y_i \in [t_{i-1}, t_i)$. Therefore, we fix $y_i = t_{i-1}$;
- this concept is called Itô stochastic integral.

Stochastic integrals

In some cases, this integral can be computed explicitly from the definition.

Example: $\int_0^t B_x\, dB_x = ?$

Letting $\Delta_i B := B_{t_i} - B_{t_{i-1}}$ and $\Delta_i t := t_i - t_{i-1}$, we have

$$\begin{aligned}\sum_i B_{t_{i-1}} (\Delta_i B) &= \sum_i \left(B_{t_{i-1}} B_{t_i} - B_{t_{i-1}}^2\right)\\ &= \frac{1}{2} \sum_i \left[\left(B_{t_i}^2 - B_{t_{i-1}}^2\right) - \left(B_{t_i} - B_{t_{i-1}}\right)^2\right]\end{aligned}$$

$$= \frac{1}{2}\left(B_t^2 - B_0^2\right) - \frac{1}{2} \sum_i \left(\Delta_i B\right)^2 = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_i \left(\Delta_i B\right)^2,$$

and

$$\int_0^t B_x\, dB_x = L^2 - \lim_{\max_i(\Delta_i t) \to 0} \sum_i B_{t_{i-1}} (\Delta_i B).$$

Stochastic integrals

Now, $\mathbb{E}\big[\sum_i (\Delta_i B)^2\big] = \sum_i \mathrm{Var}\big[\Delta_i B\big] = \sum_i \Delta_i t = t$ and

$$\mathrm{Var}\Big[\sum_i (\Delta_i B)^2\Big] = \sum_i \mathrm{Var}\big[(\Delta_i B)^2\big] = \sum_i (\Delta_i t)^2 \mathrm{Var}\big[(\mathcal{N}(0,1))^2\big]$$
$$= 2\sum_i (\Delta_i t)^2 \leq 2(\max_i \Delta_i t) \sum_i \Delta_i t = 2t(\max_i \Delta_i t) \to 0.$$

Hence, $\sum_i \big(\Delta_i B\big)^2 \to t$ in L^2 and we obtain

$$\int_0^t B_x \, dB_x = \frac{1}{2} B_t^2 - \frac{1}{2} t.$$

Remark (quadratic variation of BMs): we showed above that $\mathbb{E}\big[(\Delta_i B)^2\big] = \Delta_i t$ and that $\mathrm{Var}\big[(\Delta_i B)^2\big] = 2(\Delta_i t)^2 = o(\Delta_i t)$. This means that "$(\Delta_i B)^2$ behaves as $\Delta_i t$" (i.e., $(dB_t)^2 = dt$).

Stochastic integrals

We showed that

$$\int_0^t B_x \, dB_x = \frac{1}{2} B_t^2 - \frac{1}{2} t.$$

This result is surprising at first sight because one would expect

$$\int_0^t B_x \, dB_x = \int_0^t d\frac{(B_x)^2}{2} = \left[\frac{(B_x)^2}{2}\right]_0^t = \frac{1}{2}\left(B_t^2 - B_0^2\right) = \frac{1}{2} B_t^2.$$

This shows that the standard chain rule does not apply...

Standard chain rule

For standard integration,

$$df(g(x)) = f(g(x+dx)) - f(g(x)) = [f(g(x))]'\,dx + \frac{[f(g(x))]''}{2}(dx)^2 + \ldots$$

yields

$$f(g(t)) - f(g(0)) = \int_0^t df(g(x)) = \int_0^t [f(g(x))]'\,dx = \int_0^t f'(g(x))\,dg(x).$$

This is the standard chain rule.

For stochastic integration, the chain rule (with $f(x) = x^2$ and $g(x) = B_x$) yields

$$B_t^2 - B_0^2 = \int_0^t 2B_x\,dB_x \left(\neq B_t^2 - t \right)$$

and is therefore violated.

Itô's lemma

How to extend the chain rule?

From the quadratic variation of the BM,

$$df(B_x) = f(B_{x+dx}) - f(B_x) = f'(B_x)\,dB_x + \frac{f''(B_x)}{2}(dB_x)^2 + \dots$$
$$= f'(B_x)\,dB_x + \frac{f''(B_x)}{2}\,dx + \dots,$$

which yields

$$f(B_t) - f(B_0) = \int_0^t df(B_x) = \int_0^t f'(B_x)\,dB_x + \frac{1}{2}\int_0^t f''(B_x)\,dx.$$

This is the so-called Itô lemma.

Itô's lemma

Itô's lemma states that

$$f(B_t) - f(B_0) = \int_0^t f'(B_x)\, dB_x + \frac{1}{2}\int_0^t f''(B_x)\, dx.$$

In our example, it yields (with $f(x) = x^2$)

$$B_t^2 - B_0^2 = \int_0^t 2B_x\, dB_x + \frac{1}{2}\int_0^t 2\, dx = \int_0^t 2B_x\, dB_x + t,$$

which provides our result

$$\int_0^t B_x\, dB_x = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

Itô's lemma: extension 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ with partial derivatives $f_1, f_2, f_{11}, f_{12}, f_{22}$, etc. Then

$$\begin{aligned} df(x, B_x) &= f_1(x, B_x)\, dx + f_2(x, B_x)\, dB_x + \frac{f_{11}(x, B_x)}{2} (dx)^2 \\ &\quad + \frac{f_{22}(x, B_x)}{2} (dB_x)^2 + f_{12}(x, B_x)\, dx\, dB_x + \dots \\ &= \left[f_1(x, B_x) + \frac{f_{22}(x, B_x)}{2} \right] dx + f_2(x, B_x)\, dB_x + \dots, \end{aligned}$$

which yields

$$f(t, B_t) - f(0, B_0) = \int_0^t \left[f_1(x, B_x) + \frac{f_{22}(x, B_x)}{2} \right] dx + \int_0^t f_2(x, B_x)\, dB_x.$$

Geometric Brownian motion

Let (B_t) be a standard BM. Then a geometric BM is defined as

$$X_t = X_0 \exp\left((c - \frac{\sigma^2}{2})t + \sigma B_t\right) = f(t, B_t),$$

where $f(x, y) = X_0 \exp\left((c - \frac{\sigma^2}{2})x + \sigma y\right)$.

The previous extension of the Itô lemma yields

$$X_t - X_0 = f(t, B_t) - f(0, B_0) = \int_0^t \left[\left(c - \frac{\sigma^2}{2}\right) f(x, B_x) + \frac{\sigma^2 f(x, B_x)}{2}\right] dx$$

$$+ \int_0^t \sigma f(x, B_x)\, dB_x = c \int_0^t X_x\, dx + \sigma \int_0^t X_x\, dB_x.$$

Such a process is called an Itô process.

Geometric Brownian motion

Differentiating, we obtain

$$dX_t = cX_t\,dt + \sigma X_t\,dB_t,$$

that is,

$$\frac{X_{t+dt} - X_t}{X_t} = c\,dt + \sigma\,dB_t.$$

In words, the relative return from the asset in $[t, t + dt]$ is given by a linear trend disturbed by a stochastic noise term; c is the so-called mean rate of return and σ is the volatility (which is a measure of the riskiness of the asset).

The geometric BM should be thought of as a randomly perturbed exponential function (if $\sigma = 0$, $X_t = X_0 \exp(ct)$).

People in economics believe in exponential growth. Hence, they are quite satisfied with this model...

Itô's lemma: extension 2

Consider the process $(f(t, X_t))_{t \geq 0}$, where

$$X_t = X_0 + \int_0^t a_x \, dx + \int_0^t \sigma_x \, dB_x,$$

where (a_x) and (σ_x) are adapted to (B_x).

Then

$$f(t, X_t) - f(0, X_0) = \int_0^t \left[f_1(x, X_x) + a_x \, f_2(x, X_x) + \frac{1}{2} \sigma_x^2 \, f_{22}(x, X_x) \right] dx + \int_0^t \sigma_x \, f_2(x, X_x) \, dB_x.$$

An application: Black-Scholes formula

Consider a risky asset

$$X_t = X_0 + c \int_0^t X_x \, dx + \sigma \int_0^t X_x \, dB_x$$

and a non-risky asset (a bond) β_t (such as a bank account)

$$\beta_t = \beta_0 e^{rt};$$

here, the initial capital β_0 has been continuously compounded with a constant interest rate r. Note that we have

$$\beta_t = \beta_0 + r \int_0^t \beta_x \, dx.$$

An application: Black-Scholes formula

This leads to the concept of portfolio:

$$V_t = a_t X_t + b_t \beta_t.$$

At time t, you hold a certain amount of shares (a_t in stock and b_t in bond).

The SPs (a_t) and (b_t) are assumed to be adapted to (B_t) (then we speak of the trading strategy (a_t, b_t)).

We allow for negative values of a_t and b_t:

- $a_t < 0$ means short sale of stock.
- $b_t < 0$ means you borrow money at the bond's riskless interest rate r.

An application: Black-Scholes formula

Remarks:

- no time-dependent interest rates; no transaction costs;
- no boundedness conditions on a_t and b_t;
- you spend no money on other purposes (i.e., you do not make your portfolio smaller by consumption);
- we restrict to self-financing trading strategies, i.e.,

$$dV_t = d\big(a_t\, X_t + b_t\, \beta_t\big) = a_t\, dX_t + b_t\, d\beta_t,$$

so that

$$V_t - V_0 = \int_0^t a_x\, dX_x + \int_0^t b_x\, d\beta_x.$$

Your wealth at time t is given by your initial wealth + the capital gains from stock and bond up to time t (no extra source of money).

An application: Black-Scholes formula

In this setup, the BS formula tells what is the rational price for a European call option.

What is a European call option (ECO)?

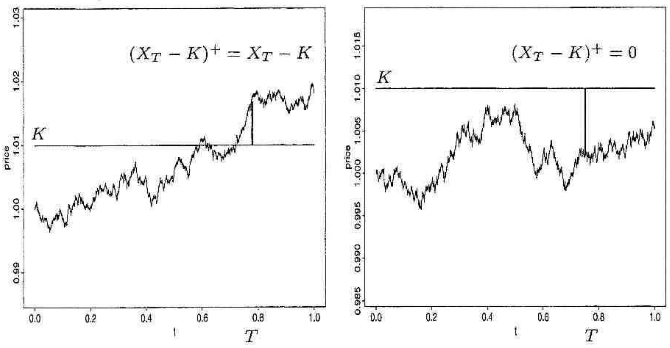
It is a ticket (you buy at time $t = 0$, say) that entitles you to buy one share of stock at a fixed price K (the exercise price or strike price) at a fixed time T (the time of maturity or time of expiration).

The holder of the option is not obliged to exercise it (it would be silly to do so if $X_T < K$!) The holder is therefore entitled to a payment of

$$(X_T - K)^+ = \max(X_T - K, 0).$$

An application: Black-Scholes formula

The value of a European call option with exercise price K at time of maturity T:



An application: Black-Scholes formula

Of course, when you purchase the call (at time $t = 0$, say), you do not know the value X_T.

Problem: how much would you be willing to pay for this ticket?

The Black-scholes-Merton rule: this rational price is the value V_0 which guarantees the same payoff (by portfolio management) as the option, namely $(X_T - K)^+$.

Remark: it can be shown that, if the price of the option is not that rational value, then there is an opportunity of arbitrage (that is, there exists a strategy which ensures an unbounded profit without any risk of loss).

An application: Black-Scholes formula

Hence, to determine this rational price, we should find V_0 satisfying

$$V_t = f(t, X_t) = a_t\, X_t + b_t\, \beta_t$$
$$V_T = f(T, X_T) = (X_T - K)^+,$$

or equivalently, letting $u(t, X_t) := f(T - t, X_t)$,

$$V_t = u(T - t, X_t) = a_t\, X_t + b_t\, \beta_t$$
$$V_T = u(0, X_T) = (X_T - K)^+.$$

We then want to determine the value of $V_0 = u(T, X_0)$.

Remark: we assume that u is smooth, which is clearly a restriction.

An application: Black-Scholes formula

Principle of the computation...

An application: Black-Scholes formula

(i) By using the 2nd extension of Itô's lemma,

$$\begin{aligned}
V_T - V_0 &= f(t, X_t) - f(0, X_0) \\
&= \int_0^t \left[f_1(x, X_x) + a_x\, f_2(x, X_x) + \frac{1}{2}\sigma_x^2\, f_{22}(x, X_x) \right] dx \\
&+ \int_0^t \sigma_x\, f_2(x, X_x)\, dB_x \\
&= \int_0^t \left[f_1(x, X_x) + cX_x\, f_2(x, X_x) + \frac{1}{2}\sigma^2 X_x^2\, f_{22}(x, X_x) \right] dx \\
&+ \int_0^t \sigma X_x\, f_2(x, X_x)\, dB_x \\
&= \int_0^t \Big[-u_1(T - x, X_x) + cX_x u_2(T - x, X_x) \\
&+ \frac{1}{2}\sigma^2 X_x^2 u_{22}(T - x, X_x) \Big] dx + \int_0^t \sigma X_x u_2(T - x, X_x)\, dB_x .
\end{aligned}$$

An application: Black-Scholes formula

(ii)

$$
\begin{aligned}
V_T - V_0 &= \int_0^t a_x \, dX_x + \int_0^t b_x \, d\beta_x \\
&= \int_0^t a_x \, (cX_x \, dx + \sigma X_x \, dB_x) + \int_0^t b_x \, d(\beta_0 e^{rx}) \\
&= \int_0^t c a_x X_x \, dx + \int_0^t \sigma a_x X_x \, dB_x + \int_0^t r b_x \beta_0 e^{rx} \, dx \\
&= \int_0^t \left(c a_x X_x + r b_x \beta_x \right) dx + \int_0^t \sigma a_x X_x \, dB_x \\
&= \int_0^t \left(c a_x X_x + r(V_x - a_x X_x) \right) dx + \int_0^t \sigma a_x X_x \, dB_x .
\end{aligned}
$$

An application: Black-Scholes formula

Now, by using the fact that

$$\int_0^t P_x^{(1)}\,dx + \int_0^t P_x^{(2)}\,dB_x = \int_0^t Q_x^{(1)}\,dx + \int_0^t Q_x^{(2)}\,dB_x$$

iff

$$\begin{cases} P_x^{(1)} = Q_x^{(1)} \\ P_x^{(2)} = Q_x^{(2)}, \end{cases}$$

one can obtain partial differential equations satisfied by $(t,x) \mapsto u(t,x)$, namely

$$u_1(t,x) = \frac{\sigma^2}{2}\,x^2 u_{22}(t,x) + r\,x\,u_2(t,x) - r\,u(t,x),$$

which has to be solved, subject to the boundary condition

$$u(0,x) = (x-K)^+.$$

An application: Black-Scholes formula

The solution of this PDE is

$$u(t,x) = x\Phi(g(t,x)) - K\,e^{-rt}\Phi(h(t,x)),$$

where Φ denotes the cdf of the standard normal distribution and

$$\begin{cases} g(t,x) = \frac{1}{\sigma\sqrt{t}}\big(\ln(x/K) + (r + \frac{\sigma^2}{2})t\big) \\ h(t,x) = g(t,x) - \sigma\sqrt{t}. \end{cases}$$

Therefore, the rational price V_0 for the option at time $t = 0$ is

$$V_0 = u(T, X_0) = X_0\Phi(g(T, X_0)) - K\,e^{-rT}\Phi(h(T, X_0)),$$

which is the Black-Scholes option pricing formula.

An application: Black-Scholes formula

Remarks:

- V_0 does not depend on c.
- $V_t = u(T-t, X_t)$ gives the value of the portfolio at time t.
- This also allows for obtaining an explicit expression for the corresponding trading strategy. More precisely,

$$a_t = u_2(T-t, X_t) \quad \text{and} \quad b_t = \frac{u(T-t, X_t) - a_t X_t}{\beta_t}.$$

  It can be shown that $a_t > 0$ for all $t \in [0, T]$. However, it may happen that $b_t < 0$ (hence, short sales of stock do not occur, but borrowing money at the bond's constant interest rate $r > 0$ may become necessary).