STATF407 Stochastic Models: second part, sheet 1

Ex 1. Let $X_1, X_2, \ldots$ be i.i.d., with $P[X_i = 1] = p$ and $P[X_i = -1] = q = 1 - p$. Let $Y_n := \sum_{i=1}^n X_i$ be a partial sum of the X_i's. Show that

- when $q = p$, $Y_n^2 - n$ is a martingale with respect to the filtration $(\mathcal{A}_n = \sigma(X_1, \ldots, X_n))_{n\geq 1}$

Ex.2. Let $X_1, X_2, \ldots$ be independent with $\mathrm{E}[X_i] = 0$ for all $i = 1, 2, \ldots$ and $Var(X_i) = \sigma_i^2$, $i = 1, 2, \ldots$ and put $S_n = \sum_{i=1}^n X_i$ and $T_n^2 = \sum_{i=1}^n \sigma_i^2$. Show that $S_n^2 - T_n^2$ is a martingale with respect to $(\mathcal{A}_n = \sigma(X_1, \ldots, X_n))_{n\geq 1}$.

Ex 3. Let $(X_n)_{n\geq 0}$ be a sequence of random variables with finite means and satisfying for $n \geq 1$,

$$E(X_{n+1}|X_0, X_1, ..., X_n) = aX_n + bX_{n-1}$$

where $0 < a, b < 1$ and $a + b = 1$.Find a value of α for which the process $(S_n)_{n\geq 0}$ defined by $S_n = \alpha X_n + X_{n-1}$, $n \geq 1$, is a martingale with respect to the sequence (X_n).

Ex 4. Proof that if (Y_n) is a martingale w.r.t. a filtration (A_n), and if (Y_n) is uniformly integrable, then

$$\lim_{n\to\infty} E[Y_n I_{T>n}] = 0$$

Ex 5. We consider the doubling strategy, for which the winnings of the gambler after n steps are

$$Y_n = \sum_{i=1}^n 2^{i-1} X_i$$

where the X_i's are i.i.d. with $\mathrm{P}[X_i = 1] = \mathrm{P}[X_i = -1] = 0.5$. Let $T = \inf n \in \mathbb{N} : X_n = 1$. Show that (Y_n) is a martingale and that T is a stopping time w.r.t. (X_n).

Ex 6. We consider the symmetric random walk, with initial wealth $Y_0 = k$ for some $k \in \mathbb{N}$. Verify that the assumptions of the optional stopping theorem are satisfied for the stopping time $T = \inf\{n \in \mathbb{N} : Y_n \in \{0, m\}\}$.

Ex 8. Let T_1 and T_2 be two stopping times with respect to a filtration $(\mathcal{F}_n)$. Show that $T_1 + T_2$, $\max(T_1, T_2)$ and $\min(T_1, T_2)$ are stopping times

Ex 9. (Doob decomposition). Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space endowed with a filtration $\mathcal{F} = (\mathcal{F}_n)$. An $\mathcal{F}$-adapted sequence A_n is an *increasing predictable process* if $A_0 = 0$ and for any $n \geq 0$, $A_n \leq A_{n+1}$ and A_{n+1} is $\mathcal{F}_n$-mesurable. Let X_n be a submartingale.

- Show that the process A_n defined by

  $$A_0 = 0, \quad A_{n+1} = A_n + \mathrm{E}[X_{n+1} - X_n \| \mathcal{F}_n]$$

  is an increasing predictable process.
- Show that the process
  $$M_n = X_n - A_n$$
  is a martingale
- Show that the decomposition

  $$X_n = M_n + A_n$$

  is unique.