STAT F 407 – STOCHASTIC PROCESSES (PART B) – JAN. 2017

NAME : **First Name :** **Section :**

- Explain all your answers.
- Please write in a readable way.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
/4	/3	/3	/3	/4	/3	/10

1.

1. State and prove the Bayes formula.
2. State and prove the Markov inequality.

2. A random variable X has density

$$f_X(x) = \begin{cases} \kappa x^3 & \text{if } x \in [0,1] \\ 0 & \text{else.} \end{cases}$$

1. Find the constant κ.
2. What is the variance of X?
3. Compute the expectation of $X^3 + 4$.

3.

Suppose a casino takes 500 successive bets at a roulette table. For simplicity assume that all players bet exactly 1 Euro on *red* or *black* (i.e. the chance to win is 18/37.)

1. What is the (approximate) probability that the bank has lost money after the 500 bets?
2. After how many of such bets can the bank be 95% sure to have gained money?

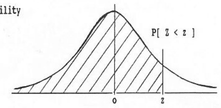
Hint: Use normal approximation. Table is attached.

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[\ Z < z\ ] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\ dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

4. Consider the homogeneous Markov chain on the states $\{1, 2, 3\}$ with transition matrix

$$P = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Draw the weighted graph associated with this Markov chain and classify the states.
2. Compute the matrices R (conditional expected number of visits) and F (conditional probability of first visit/return) and interpret them.

5.

a) What is a homogeneous Markov (continuous-time) process $(X_t)_{t\geq 0}$ on S?

b) Let $W_t = \inf\{s > 0 \,|\, X_{t+s} \neq X_t\}$ be the survival time of state X_t of a homogeneous Markov process $(X_t)_{t\geq 0}$ on S. Let $i \in S$. Show that then either

- (i) $W_t|[X_t = i] = 0$ a.s., or
- (ii) $W_t|[X_t = i] = \infty$ a.s., or
- (iii) $W_t|[X_t = i] \sim \mathrm{Exp}(\lambda_i)$ for some $\lambda_i > 0$.

6.

a) Give the definition of a standard Brownian motion $(X_t)_{t\geq 0}$.

b) Show that the process (X_t) is a martingale with respect to the natural filtration.

c) Show that the process $(Y_t)_{t\geq 0}$ defined for any $t \geq 0$ and $\theta \in \mathbb{R}$, by

$$Y_t = \exp\left(\theta X_t - \frac{\theta^2}{2}t\right)$$

is also a martingale with respect to the natural filtration.