STAT F 407 – STOCHASTIC PROCESSES – Sept. 2018

NAME : **First Name :** **Section :**

- Explain all your answers.
- Please write in a readable way.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
/4	/3	/3	/3	/4	/3	/20

1. We flip 3 fair coins (head and tail occur each with 50% chance). Let X be the number of heads.

a) Make a sketch of the distribution function of X.

b) Compute $E[X]$.

c) Compute $P(X \geq 1)$.

d) Compute $P(\text{coin 1 and coin 2 show the same side})$.

2. A random variable X has density

$$f_X(x) = \begin{cases} \kappa(x+2)^2 & \text{if } x \in [0,1] \\ 0 & \text{else.} \end{cases}$$

1. Find the constant κ.
2. What is the variance of X?
3. Compute the expectation of $X^2 + 1$.

3.

Suppose a casino takes 500 successive bets at a roulette table. For simplicity assume that all players bet exactly 1 Euro on *red* or *black* (i.e. the chance to win is 18/37.)

1. What is the (approximate) probability that the bank has lost money after the 500 bets?
2. After how many of such bets can the bank be 95% sure to have gained money?

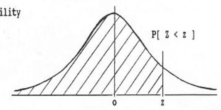
Hint: Use normal approximation. Table is attached.

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[\ Z < z\ ] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\ dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

4. Consider the homogeneous Markov chain on the states $\{1, 2, 3\}$ with transition matrix

$$P = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Draw the weighted graph associated with this Markov chain and classify the states.
2. Compute the matrices R (conditional expected number of visits) and F (conditional probability of first visit/return) and interpret them.

5. Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathrm{P}(Y_i = 1) = .5 = \mathrm{P}(Y_i = -1)$ and define $S_n := \sum_{i=1}^{n} 2^{i-1} Y_i$.

1. Show that S_n is a martingale with respect to the natural filtration $\mathcal{F}_n := \sigma(Y_1, \ldots, Y_n)$. [/1]

2. Let $T_n := \inf_n\{Y_n = 1\}$. Show that T_n is a stopping time with respect to Y_n.[/2]

6.

a) Give the definition of a Poisson process. [/1]

b) Let $\{X_t\}$ be a Poisson process with parameter $\lambda > 0$ and let W_t stand for the survival time of state X_t from t. 1) Define W_t rigorously and 2) obtain the distribution of $\sum_{i=1}^{n} W_i$. [/2]